Hyperbolic/Parabolic Divergence Cleaning for SPH

Terrence Tricco

Daniel Price

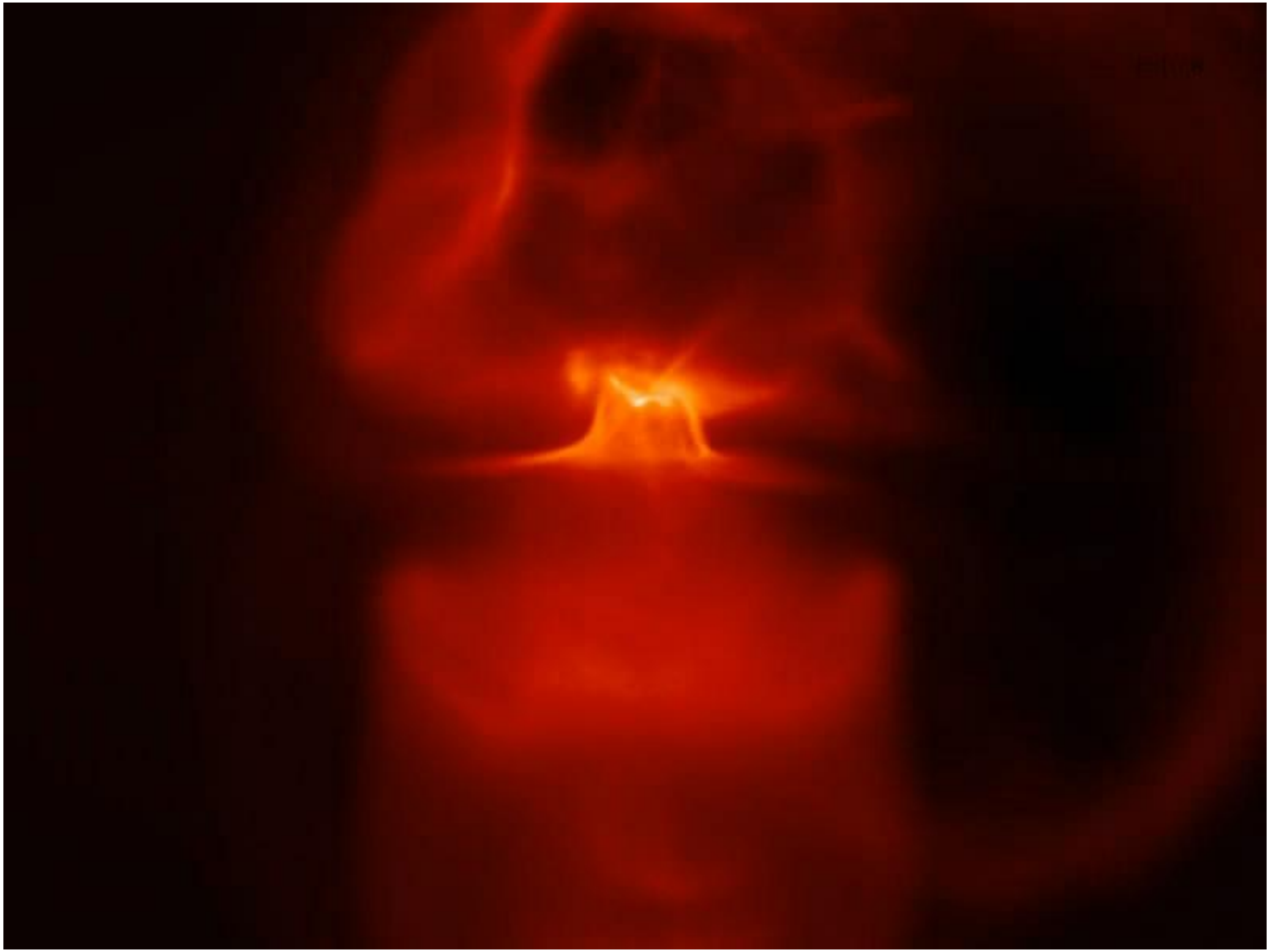
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{v} = 0$$

MoCA Monash Centre for Astrophysics

MONASH University Science

spheric ERCOFTAC



- Collapse of magnetised molecular cloud core (edge on view).
- Strong central divergence causes the protostar (and most everything else) to be ejected out of the disc.

Approaches to Control Divergence

1) Strong artificial resistivity:

$$\left(\frac{\partial \mathbf{B}}{\partial t}\right)_{\mathrm{resist}} \equiv \eta \nabla^2 \mathbf{B} = \eta \nabla \left(\nabla \cdot \mathbf{B}\right) - \eta \nabla \times \left(\nabla \times \mathbf{B}\right)$$

- Some measure of divergence control for free,
- But weakens physical field as well.

2) Euler Potentials:

$$\mathbf{B} = \nabla \alpha \times \nabla \beta$$

- Can't represent certain field configurations.

Hyperbolic/Parabolic Cleaning

Couple scalar field ψ to magnetic field:

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\psi} = -\nabla\psi$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c_h^2 \nabla\cdot\mathbf{B} - \frac{\psi}{\tau}$$

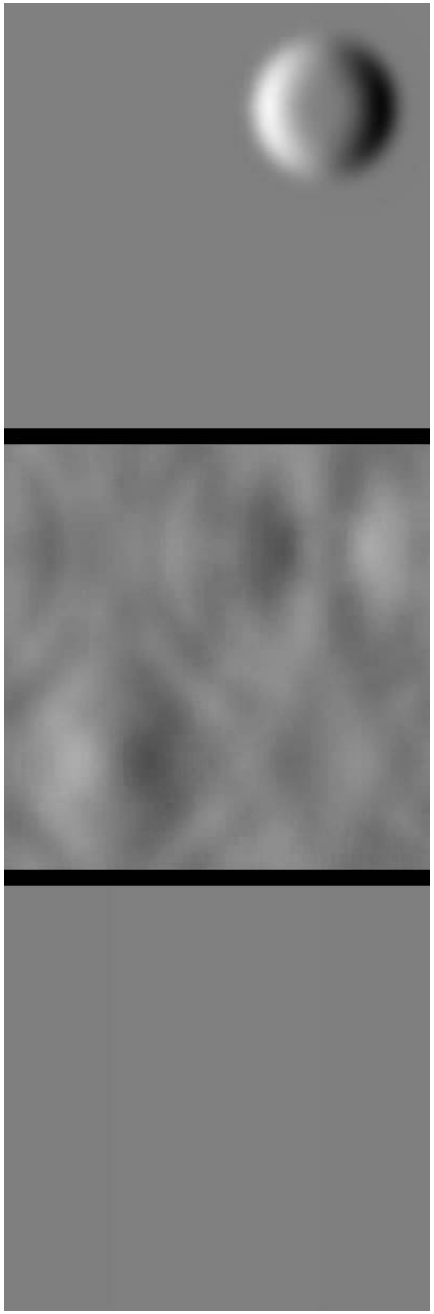
Produces damped “divergence” waves:

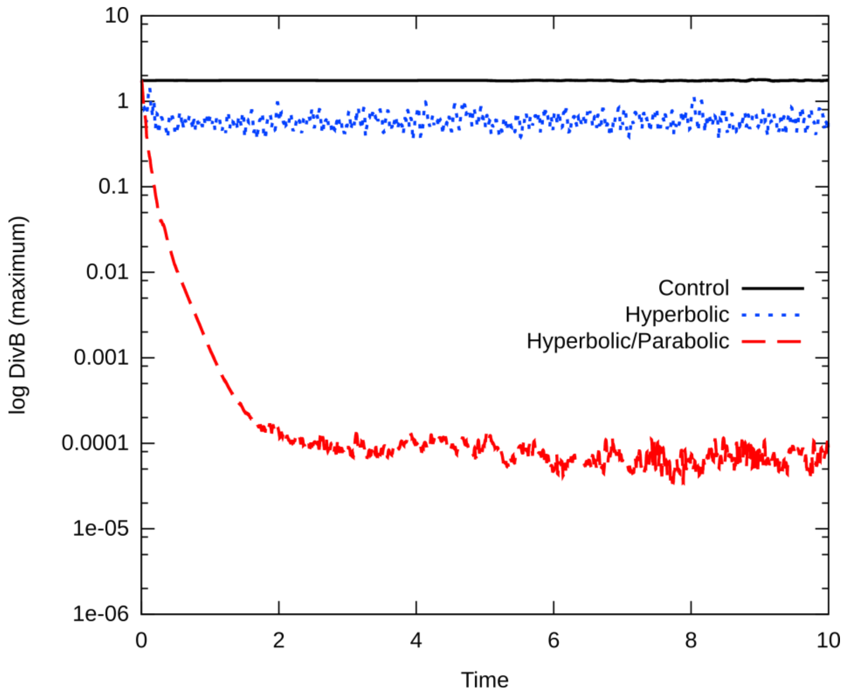
$$\frac{\partial^2(\nabla\cdot\mathbf{B})}{\partial t^2} - c_h^2 \nabla^2(\nabla\cdot\mathbf{B}) + \frac{1}{\tau}\frac{\partial(\nabla\cdot\mathbf{B})}{\partial t} = 0$$

No Cleaning

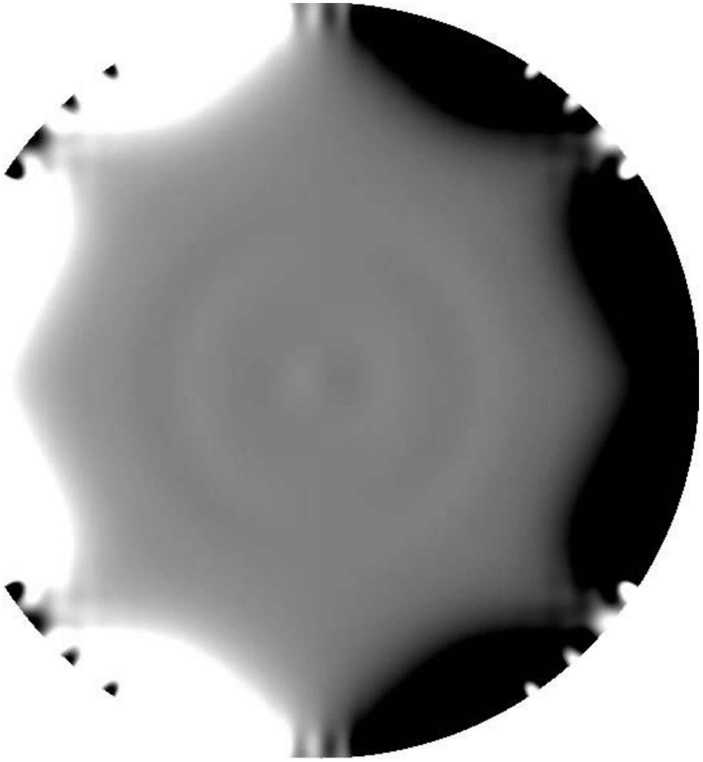
Hyperbolic

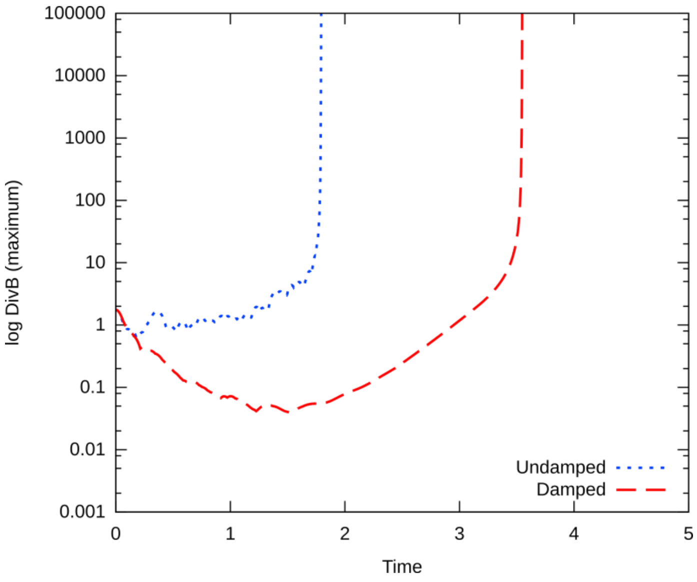
Hyperbolic/
Parabolic



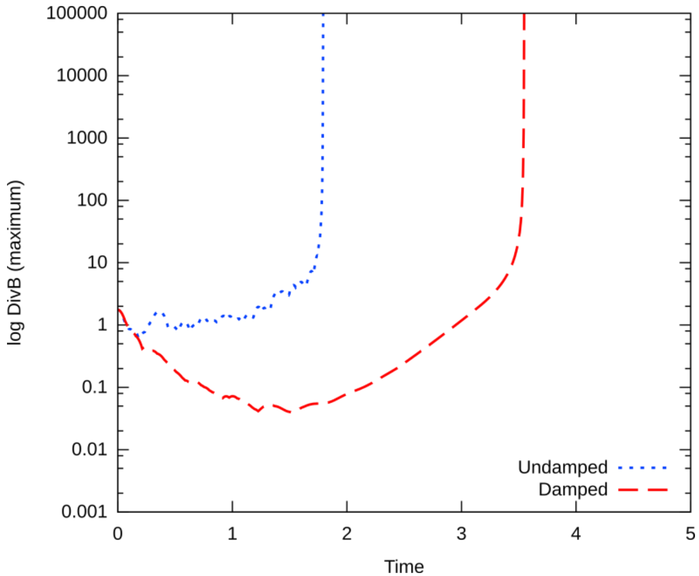


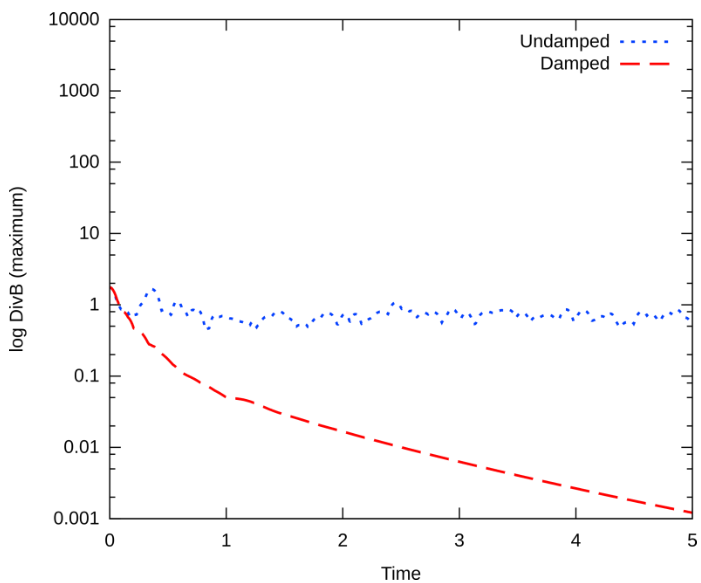
Stability Improvement





Stability Improvement





Stable Formulation

Define energy of ψ field:

$$e_\psi \equiv \frac{\psi^2}{\mu_0 \rho c_h^2}$$

Use as part of system Lagrangian:

$$L = \int \left(\tfrac{1}{2}\rho \mathbf{v}^2 - \rho u - \frac{\mathbf{B}^2}{2\mu_0} - \frac{\psi^2}{2\mu_0 c_h^2} \right) \mathrm{d}V$$

New evolution equation:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c_h^2 \nabla \cdot \mathbf{B} - \frac{\psi}{\tau} - \tfrac{1}{2}\psi \nabla \cdot \mathbf{v}$$

SPMHD Implementation

- Define divergence operator as

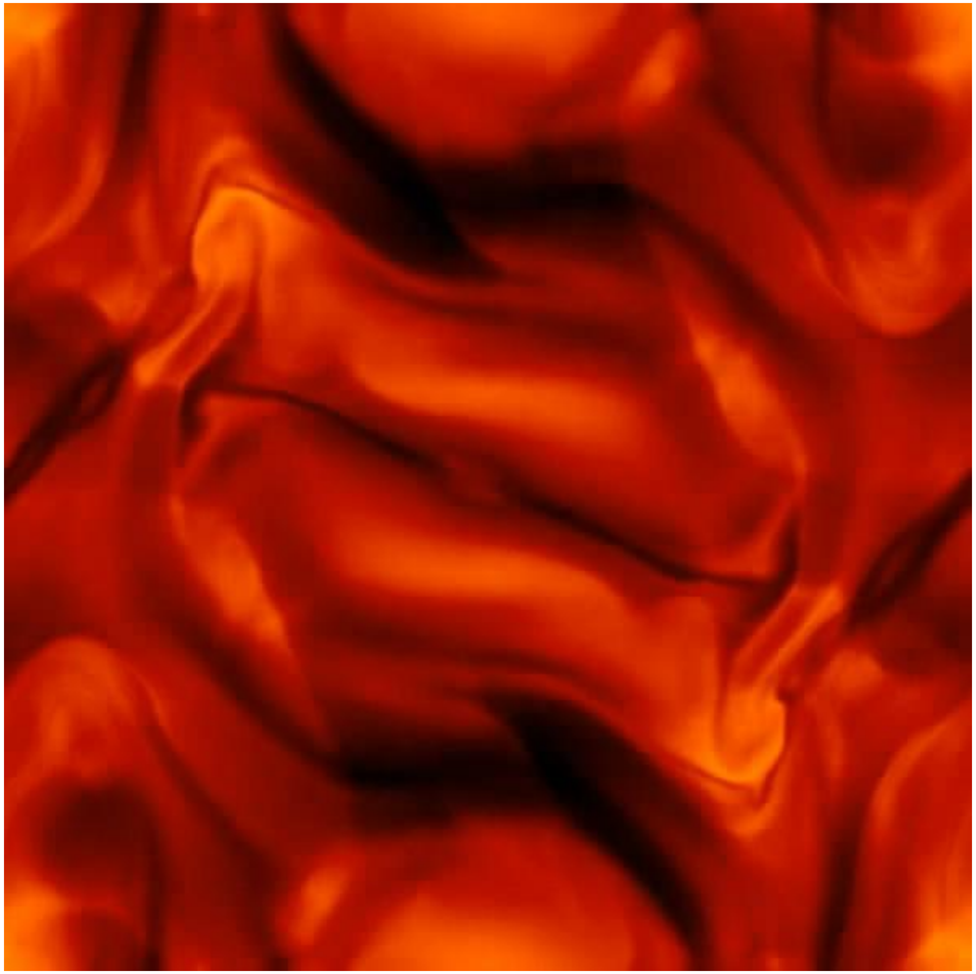
$$(\nabla \cdot \mathbf{B})_a = -\frac{1}{\Omega_a \rho_a} \sum_b m_b \left(\mathbf{B}_a - \mathbf{B}_b\right) \cdot \nabla_a W_{ab}(h_a)$$

- To conserve energy, the gradient operator for ψ **must** be;
- These two operators form a conjugate pair

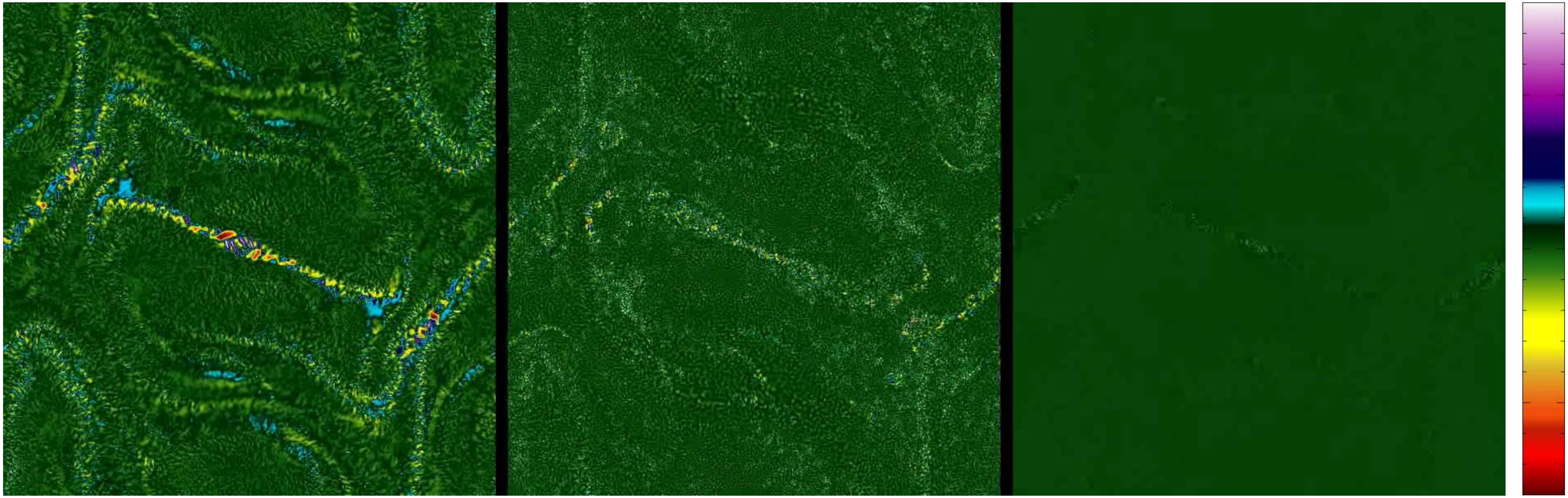
$$\left(\frac{\mathrm{d}\mathbf{B}_a}{\mathrm{d}t}\right)_\psi = -\rho_a \sum_b m_b \left[\frac{\psi_a}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{\psi_b}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b)\right]$$

Orszag-Tang Vortex

- Multiple interacting classes of shocks.
- Compare results for
  - resistivity,
  - Euler potentials,
  - divergence cleaning.

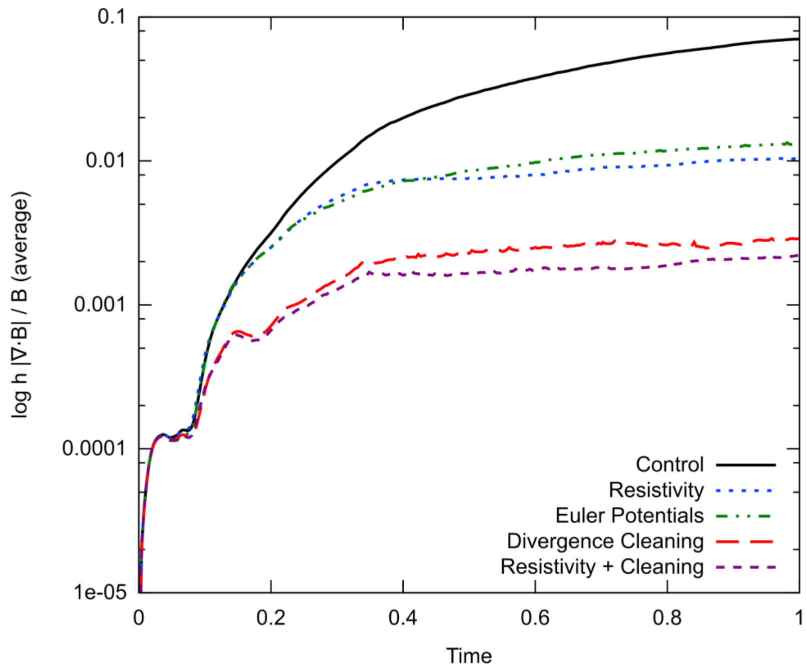


Orszag-Tang: $\nabla \cdot B$

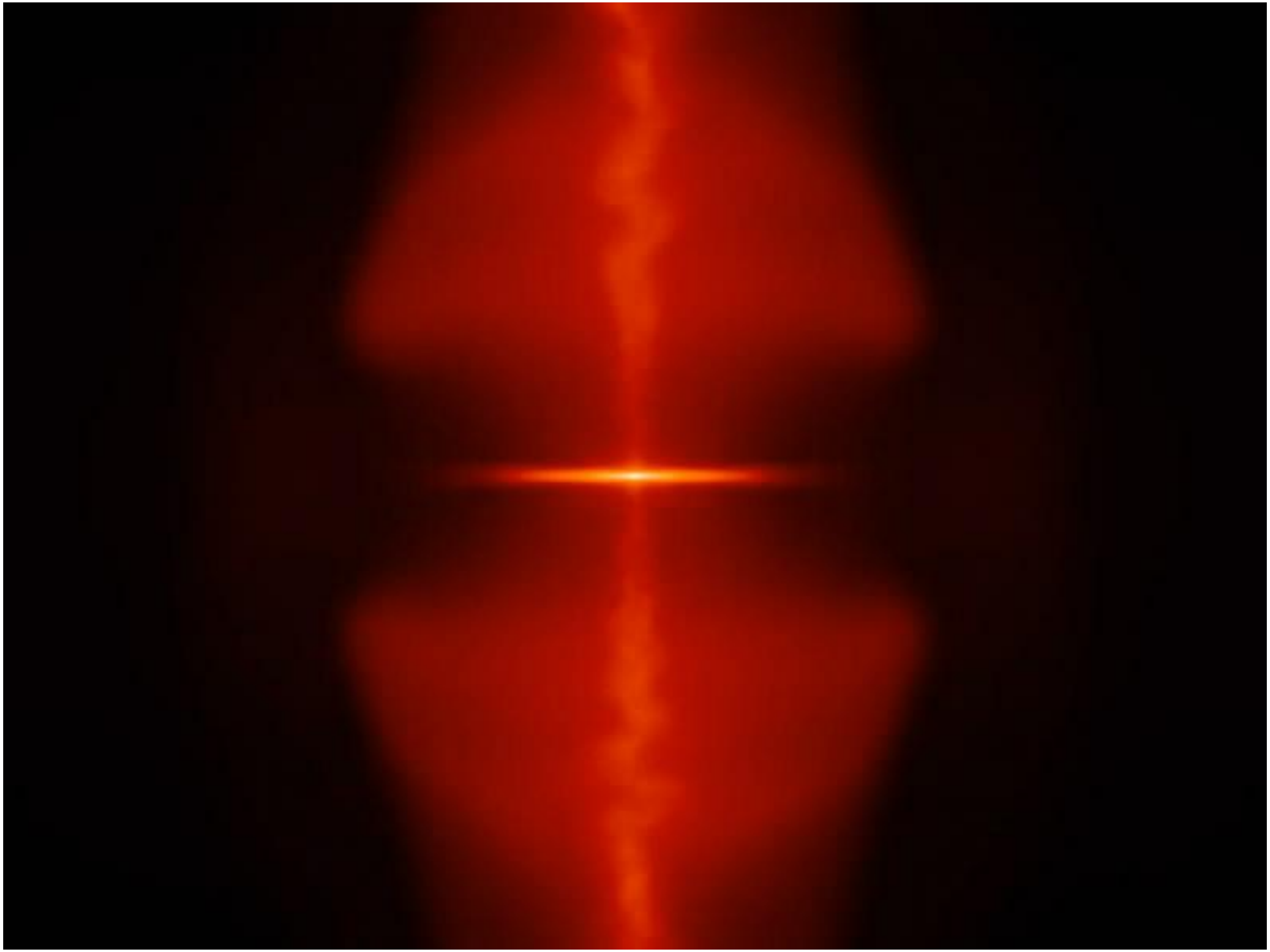


Resistivity Euler Potentials Divergence Cleaning

Orszag-Tang: Comparisons



Average divergence in the Orszag-Tang vortex. Divergence cleaning provides an order of magnitude improvement over resistivity or Euler potentials.



- With divergence cleaning, collapse remains stable.
- Momentum conservation improved by 2 orders of magnitude.
- Emergence of magnetic propelled jet!

Price, Tricco, Bate, MNRAS Letters (2012)

Cleaning Divergence of Velocity Field

- Change continuum equations to per unit density
- Still forms damped wave equation

continuum

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c_h^2 \rho \nabla \cdot \mathbf{v} - \frac{\psi}{\tau}$$

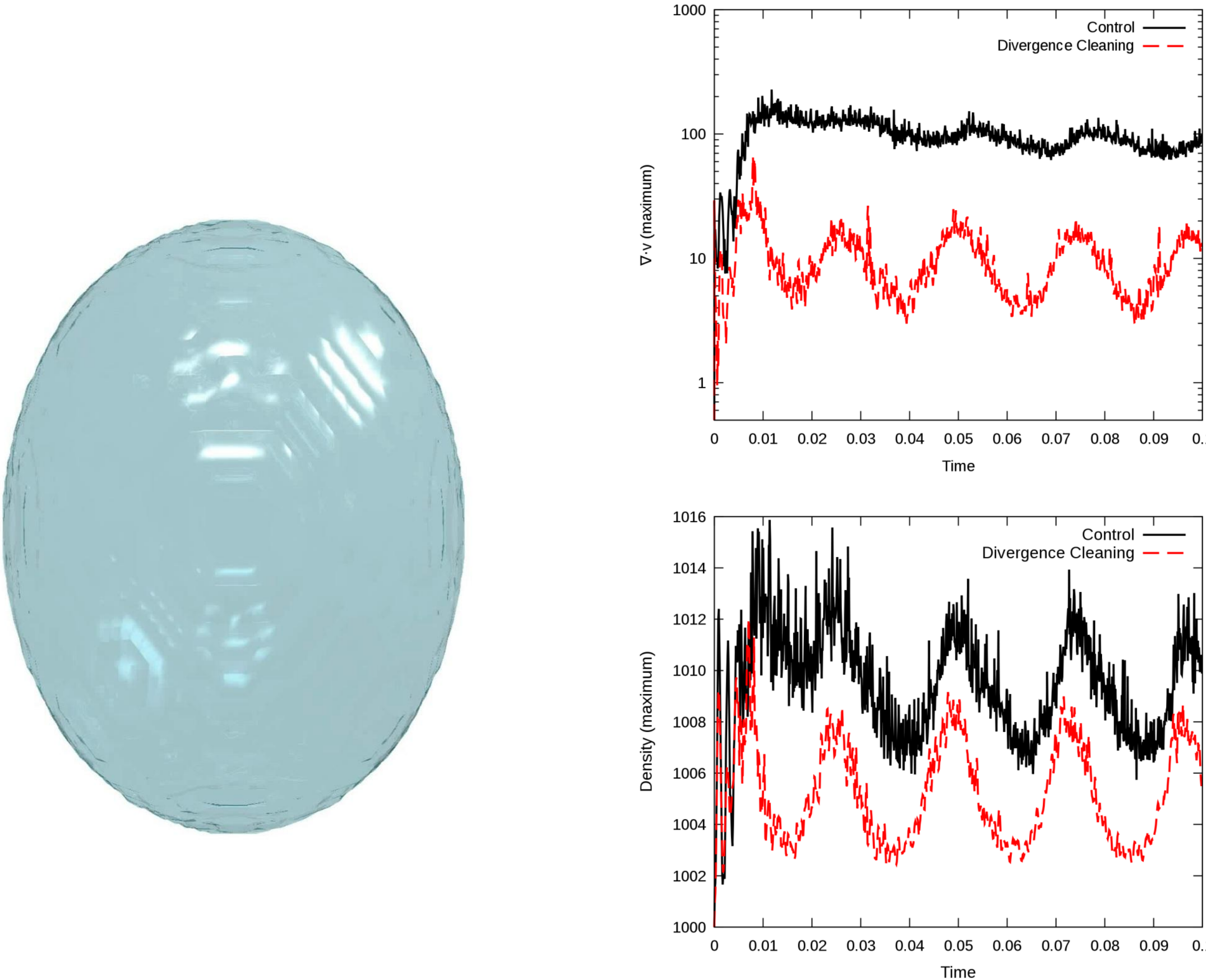
$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla \psi}{\rho}$$

SPH

$$\nabla \cdot \mathbf{v}_a = -\frac{1}{\rho_a} \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab}.$$

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_b m_b \left(\frac{\psi_a}{\rho_a^2} + \frac{\psi_b}{\rho_b^2} \right) \nabla_a W_{ab}$$

- Derivative operators again form conjugate pair
- Conserves energy, momentum



Control
Divergence Cleaning
∇·v (maximum)
Time
1000
100
10
1
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
Control
Divergence Cleaning
Density (maximum)
Time
1016
1014
1012
1010
1008
1006
1004
1002
1000
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

Conclusions

- Conservative SPH implementation of Hyperbolic/parabolic divergence cleaning:
  - Conserves energy & momentum,
  - Numerically stable at free boundaries.
- For MHD:
  - No drawbacks, unlike resistivity or Euler potentials,
  - ~order of magnitude reduction in average divergence,
  - ~2 orders of magnitude improvement in momentum conservation.
- For weakly compressible fluids:
  - Reduces density errors by ½ on simple problems,
  - Requires more testing, especially with boundary particles.