One Step Forward, Two Steps Back: Vector Potential in SPMHD

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Smoothed Particle Magnetohydrodynamics

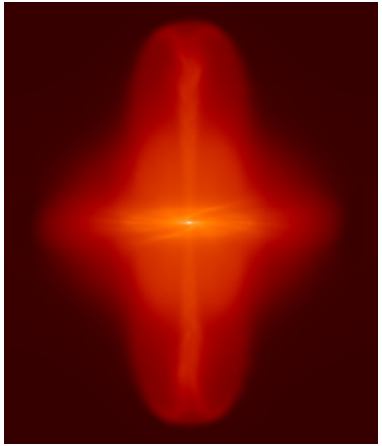
- SPMHD simulates magnetohydrodynamics (magnetized fluids).
- Magnetic field is coupled to the hydrodynamic equations.

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = \frac{1}{\rho}\frac{\partial S^{ij}}{\partial x^j} \qquad S^{ij} = -\delta^{ij}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{B^i B^j}{\mu_0}$$

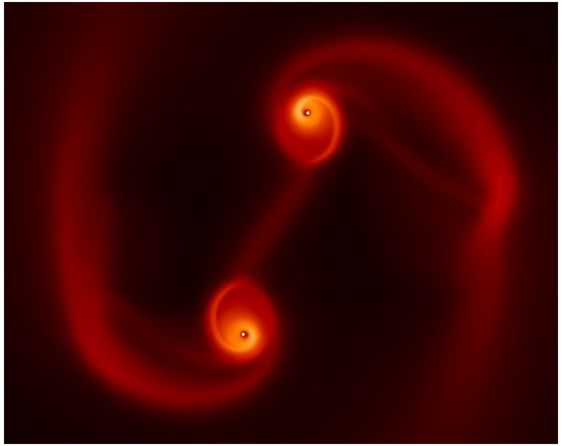
$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = -\left(\mathbf{B}\cdot\nabla\right)\mathbf{v} + \mathbf{B}\left(\nabla\cdot\mathbf{v}\right)$$

$$\nabla\cdot\mathbf{B} = 0$$

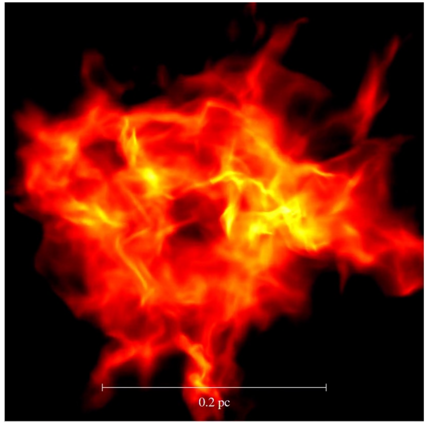
Star Formation



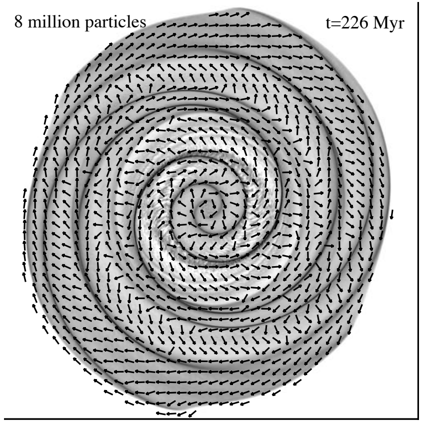
Binary Star Formation



Star Cluster Formation

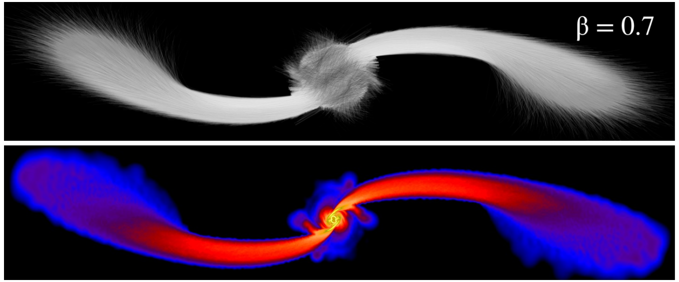


Milky Way

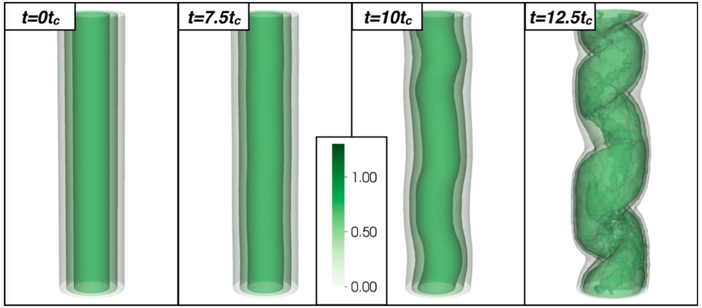


Price, **Tricco** & Bate (2012); Wurster, Price, Bate (2016); Liptai et al (2017); Dobbs, Price, Pettitt, Bate, **Tricco** (2016); Bonnerot et al (2017); Vela Vela et al (2019); **Tricco** Price & Federrath (2016)

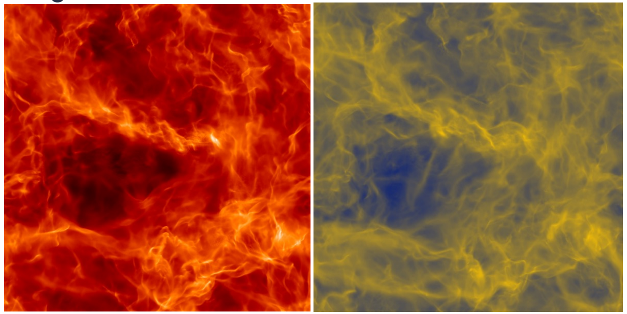
Tidal Disruption Events



Magnetically Confined Plasmas



Magnetized Turbulence



Divergence-Free Condition

- The primary challenge in simulating magnetized fluids with SPH is the divergence-free condition.

$$\nabla \cdot \mathbf{B} = 0$$

- For the past decade, SPMHD simulations have used **constrained hyperbolic divergence cleaning** (Tricco, Price 2012; Tricco, Price, Bate 2016).

- Constrained cleaning typically provides a **10x reduction in divergence error**, and keeps average **error around the ~1% level**.

Goal: Create an exactly divergence-free SPMHD

Vector Potential

- Formulating the magnetic field in terms of the **Vector Potential** is a logical choice.

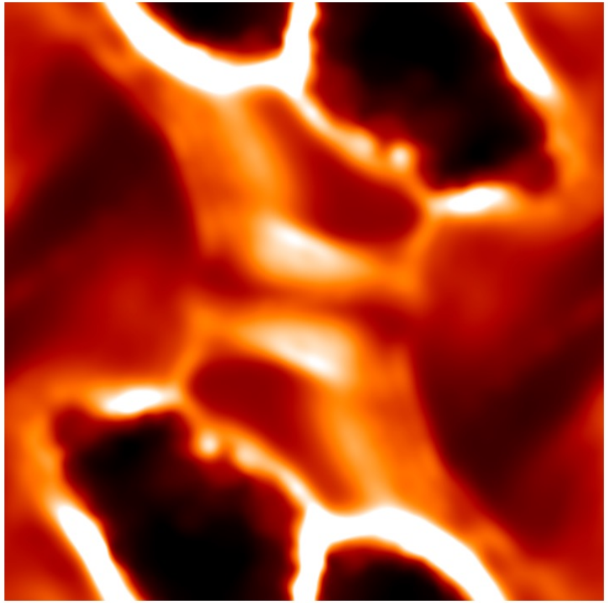
$$\mathbf{B} = \nabla \times \mathbf{A}$$

- The divergence of the curl is zero.
- Guaranteed to create a divergence-free magnetic field by construction.

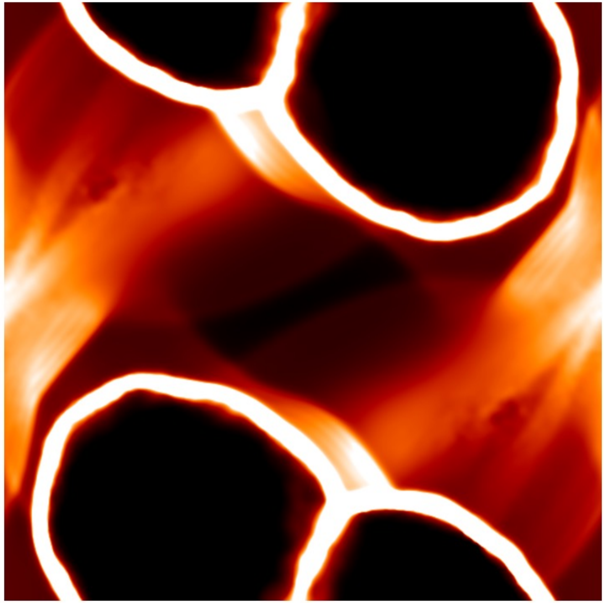
- Price (2010) investigated the vector potential, finding **severe numerical issues.**

Orszag-Tang Vortex with Vector Potential

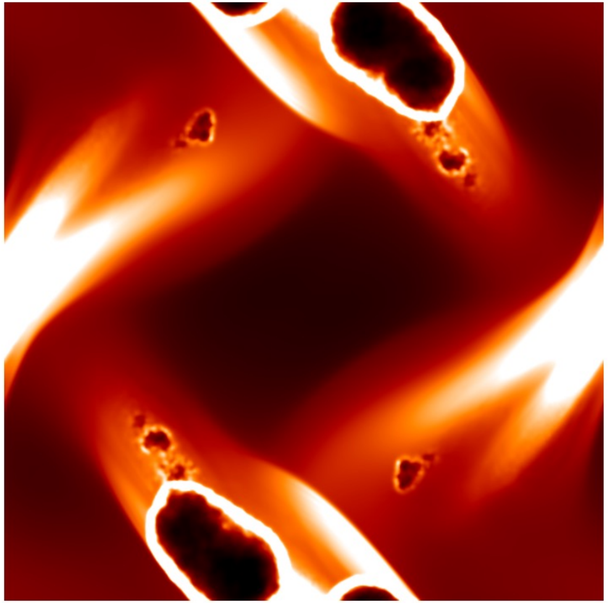
nx = 64 particles, t = 0.5



nx = 128 particles, t = 0.3



nx = 256 particles, t = 0.1975



Price (2010) Formulation

- Start with the induction equation and use $\mathbf{B} = \nabla \times \mathbf{A}$.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \longrightarrow \frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{A}) + \nabla \phi,$$

- Choose the gauge $\phi = -\mathbf{v} \cdot \mathbf{A}$ (constant of integration) yields the Galiliean invariant discretised equations:

$$\frac{\mathrm{d}A_a^i}{\mathrm{d}t} = \frac{A_a^j}{\Omega_a \rho_a} \sum_b m_b \left(v_a^j - v_b^j \right) \frac{\partial W_{ab}(h_a)}{\partial x_a^i}$$

A New Hope?

- Our new approach is to express the vector potential in **integral form**.

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{A}) \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{A}\mathrm{d}V = \int_v \mathbf{v} \times \mathbf{B}\mathrm{d}V + \int_{\partial V} \mathbf{A}\mathbf{v} \cdot \mathrm{d}S$$

$$= \int_v \mathbf{v} \times \mathbf{B}\mathrm{d}V + \int_V \nabla_j (\mathbf{A}\mathbf{v}^j)\mathrm{d}V.$$

- Which discretized into SPMHD using $\phi = -\mathbf{v} \cdot \mathbf{A}$ yields

$$\frac{\mathrm{d}A_a^i}{\mathrm{d}t} = \frac{A_a^j}{\Omega_a \rho_a} \sum_b m_b \left(v_a^j - v_b^j\right) \frac{\partial W_{ab}(h_a)}{\partial x_a^i} + \frac{1}{\Omega_a \rho_a} \sum_b m_b \left(A_a^i - A_b^i\right) \left[\left(v_a^j - v_b^j\right) \frac{\partial W_{ab}(h_a)}{\partial x_a^j}\right]$$

Comparison

- Price (2010) formulation

$$\frac{\mathrm{d}A_a^i}{\mathrm{d}t} = \frac{A_a^j}{\Omega_a \rho_a} \sum_b m_b \left(v_a^j - v_b^j\right) \frac{\partial W_{ab}(h_a)}{\partial x_a^i}$$

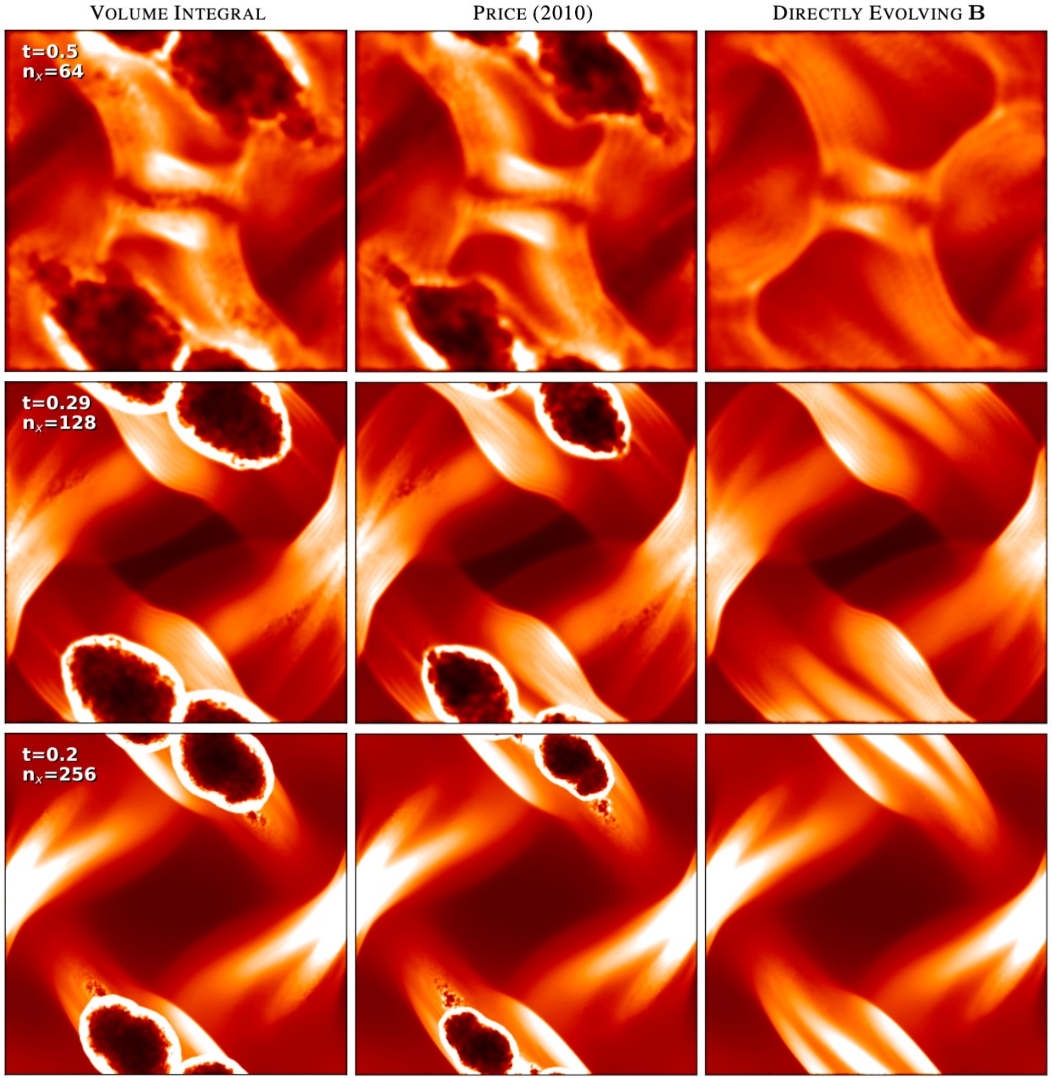
- New integral formulation

$$\frac{\mathrm{d}A_a^i}{\mathrm{d}t} = \frac{A_a^j}{\Omega_a \rho_a} \sum_b m_b \left(v_a^j - v_b^j\right) \frac{\partial W_{ab}(h_a)}{\partial x_a^i} + \frac{1}{\Omega_a \rho_a} \sum_b m_b \left(A_a^i - A_b^i\right) \left[\left(v_a^j - v_b^j\right) \frac{\partial W_{ab}(h_a)}{\partial x_a^j}\right]$$

nx = 64 particles, t = 0.5

nx = 128 particles, t = 0.29

nx = 256 particles, t = 0.2



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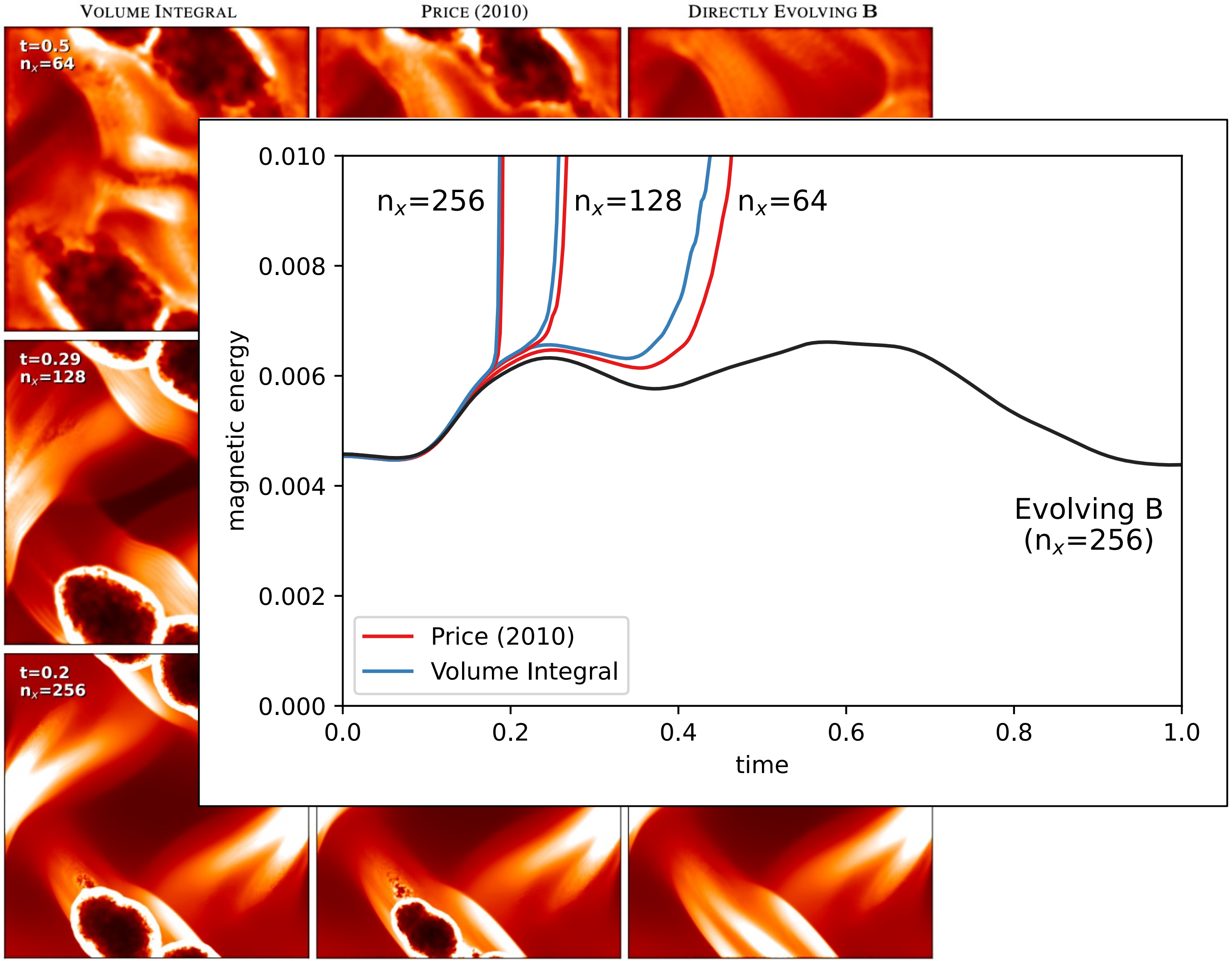

Vector Potential in SPMHD

- The integral vector potential formulation is still highly numerically unstable.

- **Why?**
- One issue is that the magnetic field is evolved using **A**, but the equations of motion use **B**.
- This mismatch of variables does not conserve energy, leading to exponential energy growth and numerical instability.

A Fully Conservative Formulation

- Enforcing energy conservation can be achieved by solving for the equations of motion from the Lagrangian (specified in terms of **A**, not **B**).

$$
\begin{aligned}
\frac{\mathrm{d}v_a^i}{\mathrm{d}t} = &-\sum_b m_b \left[\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right] \frac{\partial W_{ab}}{\partial x_a^i} \\
&+ \frac{3}{2\mu_0} \sum_b m_b \left[\frac{B_a^2}{\rho_a^2} + \frac{B_b^2}{\rho_b^2}\right] \frac{\partial W_{ab}}{\partial x_a^i} \\
&- \frac{1}{\mu_0} \epsilon_{jkl} \sum_b m_b (A_a^k - A_b^k) \left[\frac{B_a^j}{\rho_a^2} + \frac{B_b^j}{\rho_b^2}\right] \frac{\partial^2 W_{ab}}{\partial x_a^i \partial x_a^l} \\
&- \sum_b m_b \left[\frac{A_a^i}{\rho_a^2} J_a^k + \frac{A_b^i}{\rho_b^2} J_b^k\right] \frac{\partial W_{ab}}{\partial x_a^k} \\
&- \sum_b m_b (A_a^k - A_b^k) \left[\frac{J_a^k}{\rho_a^2} + \frac{J_b^k}{\rho_b^2}\right] \frac{\partial W_{ab}}{\partial x_a^i}
\end{aligned}
$$

→ Pressure gradient

} Equivalent to Price (2010) Formulation

→ Additional term from integral approach

A Fully Conservative Formulation

- Most problematic is that the equations of motion **do not represent the MHD equations of motion in the continuum limit!**
- Continuum equations derive from $\mathbf{J} \times \mathbf{B}$ where $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x^i} + \frac{1}{\rho}\left[J^j \frac{\partial A^j}{\partial x^i} - J^j \frac{\partial A^i}{\partial x^j}\right]$$

- The integral approach equations of motion are equivalent to (note the spurious factor of 3)

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x^i} + \frac{1}{\rho}\left[3J^j \frac{\partial A^j}{\partial x^i} - J^j \frac{\partial A^i}{\partial x^j}\right]$$

Summary

- An **integral approach** was developed to model the magnetic vector potential.
- Key results:
  - A correction term to the Price (2010) vector potential evolution equation is introduced. Results are **not improved**.
  - The conservative equations of motion were derived, which includes a correction term to the Price (2010) approach.
  - These equations of motion **do not represent** the MHD equations.
- A truly divergence-free SPMHD solution continues to remain elusive.

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