Fully Conservative Hyperbolic Divergence Cleaning

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UNIVERSITY OF TORONTO

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..and dusty turbulence!

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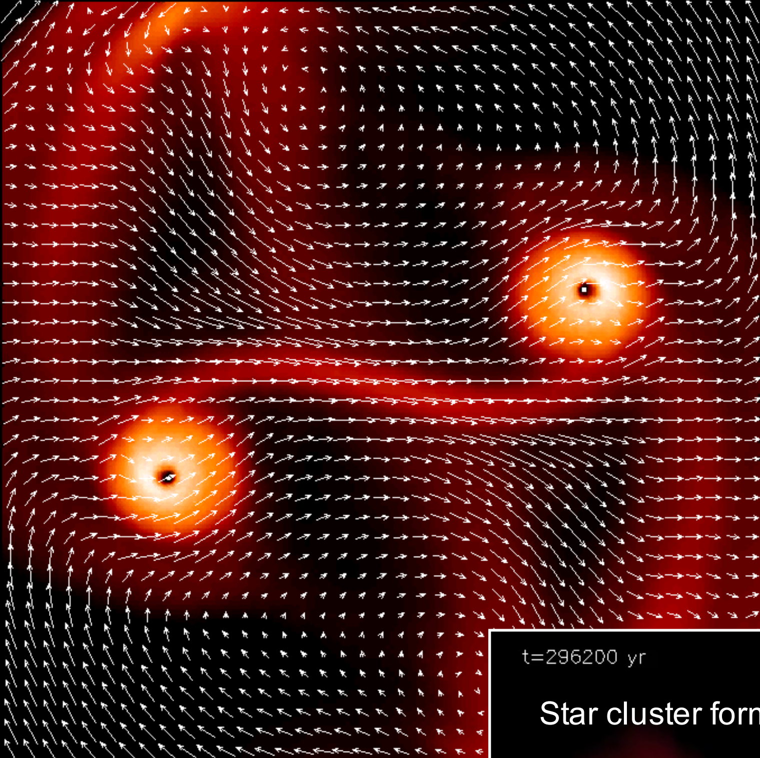
Smoothed Particle Hydrodynamics (SPH)

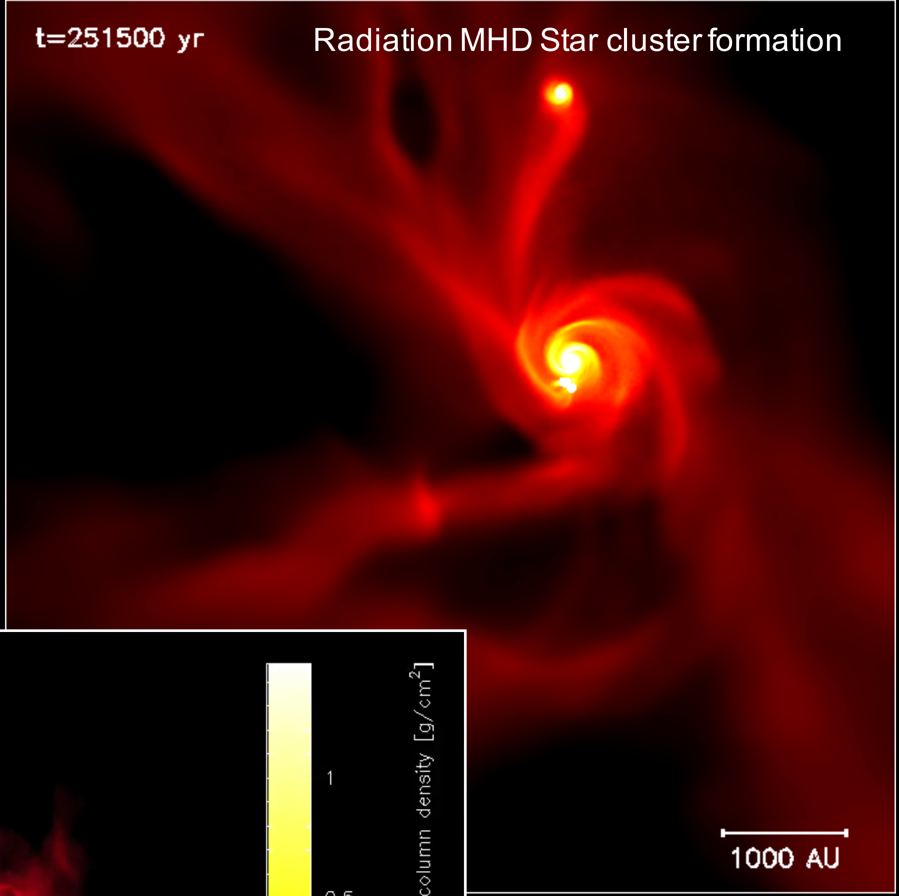
- Lagrangian particle based method for solving equations of hydrodynamics
- Advantages of SPH for astrophysics:
    1. Natively adaptive resolution wrt density
    2. Natural coupling with gravity
    3. Exact conservation of mass, energy, entropy, linear & angular momentum
    4. Exact advection
    5. Easy to model complex geometries
    6. Courant timestep free of fluid velocity

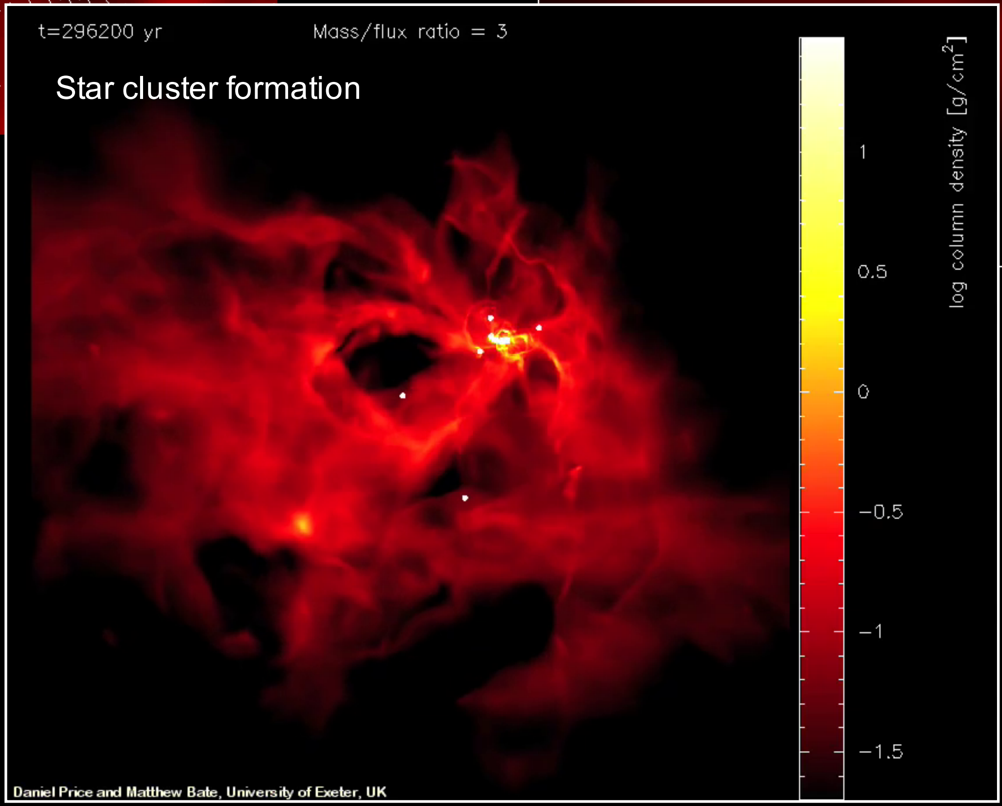
"*.. magnetic fields may be included without difficulty..*"

Gingold & Monaghan (1977)
"Smoothed particle hydrodynamics - Theory and application to non-spherical stars"



Binary star formation





Price & Bate (2007, 08, 09)

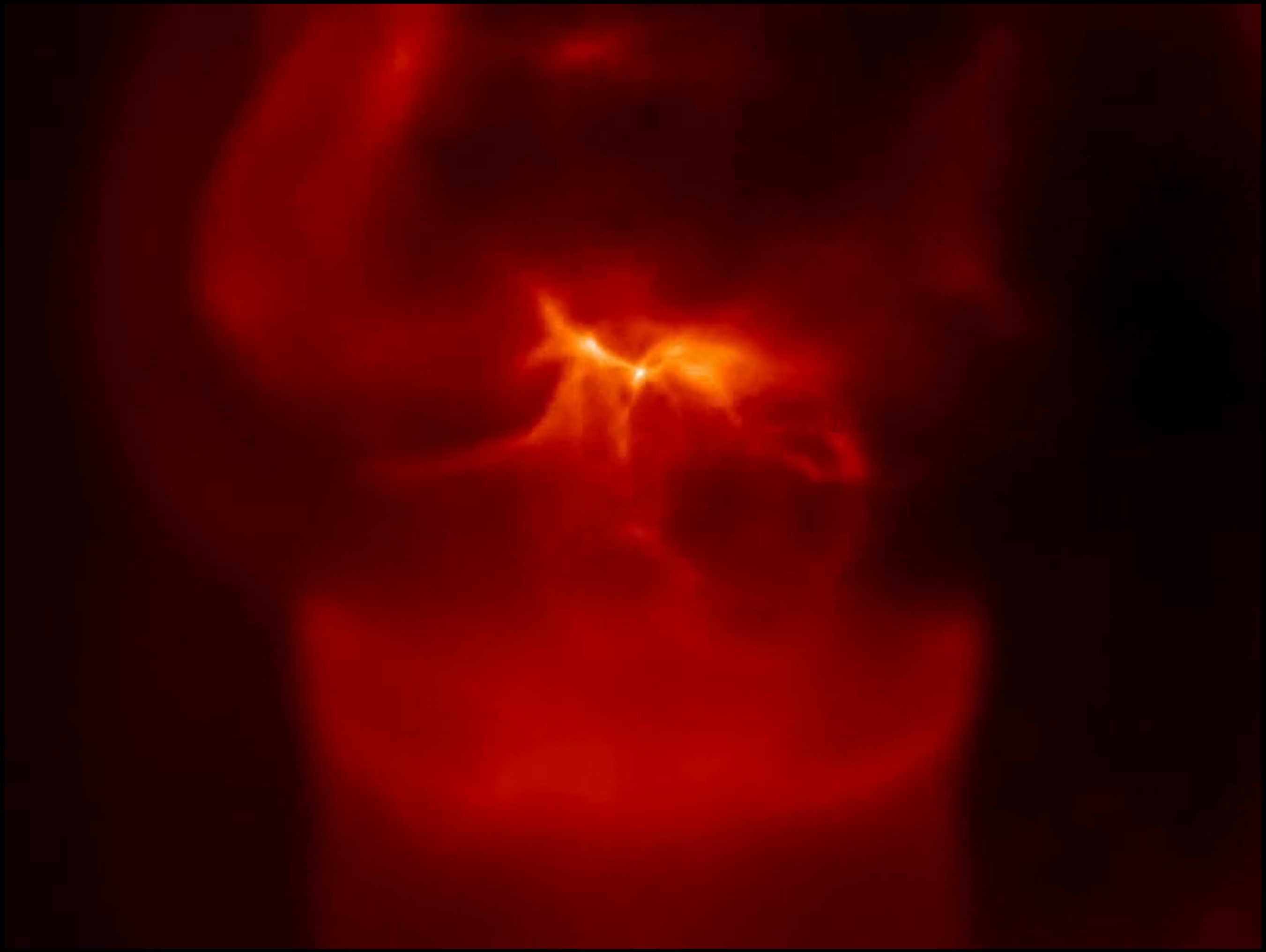
Euler Potentials

- Define the magnetic field in terms of the Euler Potentials:

$$\boldsymbol{B} = \nabla\alpha \times \nabla\beta$$

- α and β are passive scalars
- Guaranteed divergence-free magnetic field
- Works really well to provide stable numerical solutions

- But *cannot model dynamo action* nor *winding motions*
  - Incompatible with non-ideal MHD! (Brandenburg 2010)
  - Cannot do MHD turbulence! (huge problem for most of astrophysics!)



MHD; $1M_\odot$ prestellar core; B_z = 160 μG SPMHD ~ 2010

Hyperbolic Divergence Cleaning

- Directly evolve the magnetic field with induction equation
- Clean divergence errors using hyperbolic / parabolic cleaning (Dedner et al 2002):

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \psi \qquad \frac{\partial \psi}{\partial t} = -c_{\rm h}^2 \nabla \cdot \boldsymbol{B} - \frac{\psi}{\tau}$$

- Produces damped “divergence” waves:

$$\frac{\partial^2 (\nabla \cdot \boldsymbol{B})}{\partial^2 t} - c_{\rm h}^2 \nabla^2 (\nabla \cdot \boldsymbol{B}) + \frac{1}{\tau} \frac{\partial (\nabla \cdot \boldsymbol{B})}{\partial t} = 0$$

- Wave cleaning speed: $c_{\rm h}$
- Damping timescale: $\tau = \Delta x / \sigma c_{\rm h}$

Computationally inexpensive and easy to code!

“Constrained” Divergence Cleaning

- Implement divergence cleaning in a Lagrangian, SPH way
- Consider energy content of ψ field

$$e_\psi \equiv \frac{\psi^2}{2\mu_0 \rho c_\mathrm{h}^2} \qquad E = \int \left[\frac{1}{2}\boldsymbol{v}^2 + u + \frac{1}{2}\frac{\boldsymbol{B}^2}{\mu_0 \rho} + e_\psi \right] \rho \mathrm{d}V$$

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- Constrain discretised equations to conserve total energy (B + e_ψ)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\frac{\boldsymbol{B}}{\mu_0 \rho} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_\psi + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi^2}{2\mu_0 \rho c_\mathrm{h}} \right) \right] \rho \mathrm{d}V \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}t} = \sum_a m_a \left[\frac{\boldsymbol{B}_a}{\mu_0 \rho_a} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_\psi + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi_a^2}{2\mu_0 \rho_a c_\mathrm{h}} \right) \right]$$

"Constrained" Divergence Cleaning

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- Constrain discretised equations to conserve total energy (B + e_ψ)

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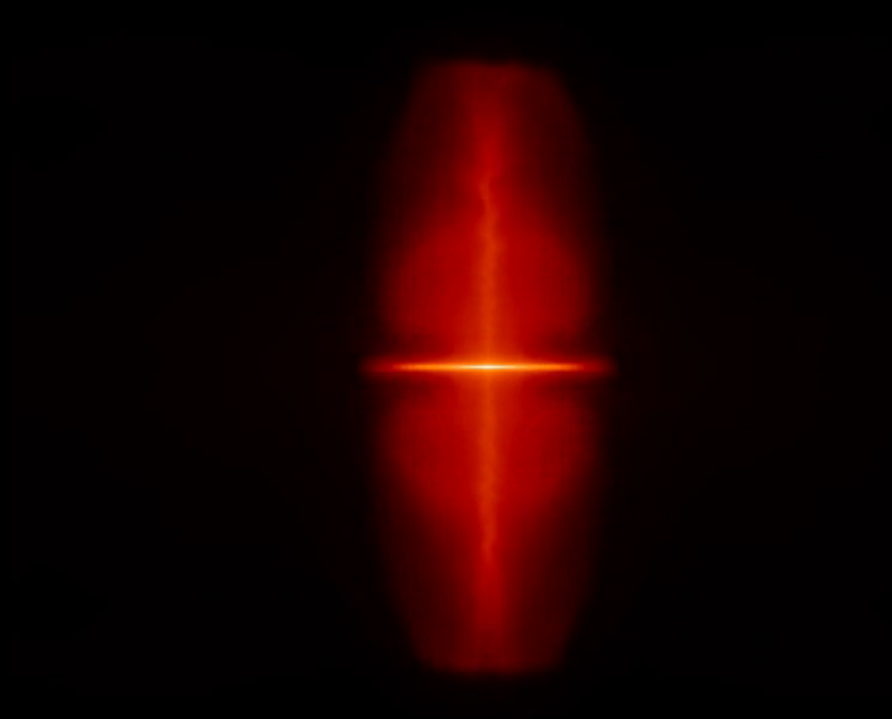
- Requires conjugate SPH derivative operators
- Obtain conservation properties inherent to SPH (*'constrained'*)
- Guaranteed to only ever decrease divergence error!

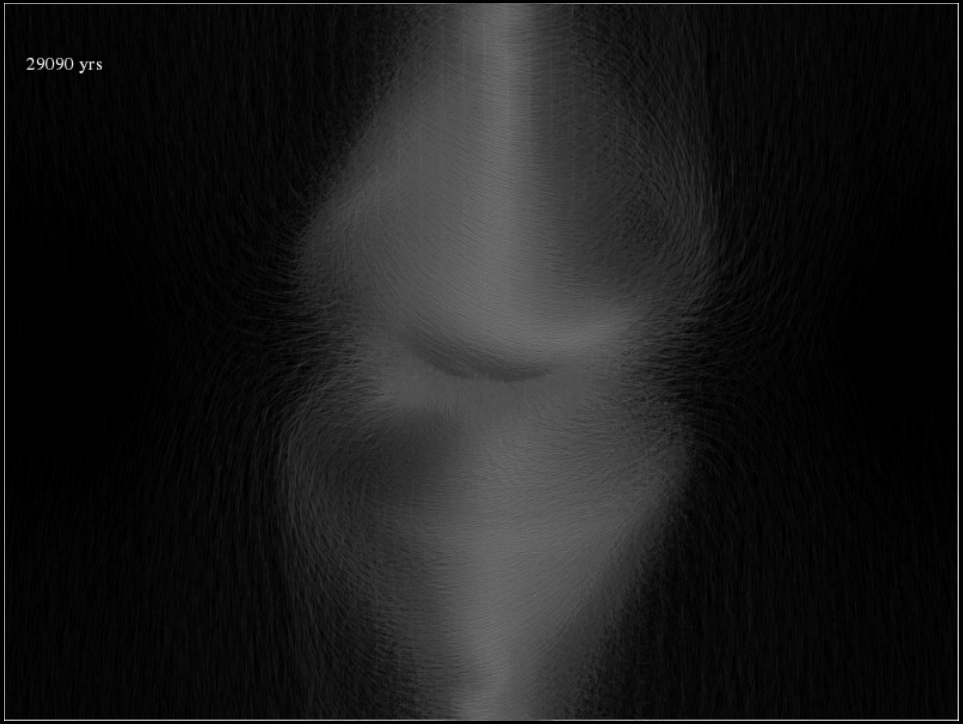
$$\frac{\mathrm{d}\boldsymbol{B}_a}{\mathrm{d}t} = -\rho_a \sum_b m_b \left(\frac{\psi_a}{\rho_a^2} + \frac{\psi_b}{\rho_b^2} \right) \nabla_a W_{ab} \qquad \frac{\mathrm{d}\psi_a}{\mathrm{d}t} = \frac{c_\mathrm{h}^2}{\rho_a} \sum_b m_b (\boldsymbol{B}_a - \boldsymbol{B}_b) \cdot \nabla_a W_{ab} - \frac{\psi_a}{\tau}$$

*Forces particular choice of derivative operators for $\nabla\psi$ and $\nabla \cdot$**B**!*

Collimated Jets from the First Core

Price, Tricco & Bate (2012)





MHD; 1 $M_\odot$ prestellar core; B_z ~ 160 μG; 5 AU sink particle radius

Fully Conservative Divergence Cleaning

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\frac{\boldsymbol{B}}{\mu_0 \rho} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_{\psi} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi^2}{2\mu_0 \rho c_{\mathrm{h}}} \right) \right] \rho \mathrm{d}V = 0$$

Fully Conservative Divergence Cleaning

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- Add $\nabla \cdot \boldsymbol{v}$ term to dψ/dt to account for compression/rarefaction

Fully Conservative Divergence Cleaning

What do you do for this??

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\frac{\boldsymbol{B}}{\mu_0 \rho} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_\psi + \frac{\psi}{\mu_0 \rho c_\mathrm{h}^2} \frac{\mathrm{d}\psi}{\mathrm{d}t} - \frac{\psi^2}{2\mu_0 \rho^2 c_\mathrm{h}^2} \frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{\psi^2}{\mu_0 \rho c_\mathrm{h}^3} \frac{\mathrm{d}c_\mathrm{h}}{\mathrm{d}t} \right] \rho \mathrm{d}V = 0$$

- Add $\nabla \cdot \boldsymbol{v}$ term to dψ/dt to account for compression/rarefaction

Fully Conservative Divergence Cleaning

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\frac{\boldsymbol{B}}{\mu_0 \rho} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_\psi + \frac{\psi}{\mu_0 \rho c_\mathrm{h}^2} \frac{\mathrm{d}\psi}{\mathrm{d}t} - \frac{\psi^2}{2\mu_0 \rho^2 c_\mathrm{h}^2} \frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{\psi^2}{\mu_0 \rho c_\mathrm{h}^3} \frac{\mathrm{d}c_\mathrm{h}}{\mathrm{d}t} \right] \rho \mathrm{d}V = 0$$

- Add $\nabla \cdot \boldsymbol{v}$ term to dψ/dt to account for compression/rarefaction
- Evolve ψ / c_h instead of ψ to account for time-varying c_h

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\frac{\boldsymbol{B}}{\mu_0 \rho} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_\psi + \frac{\psi}{\mu_0 \rho c_\mathrm{h}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi}{c_\mathrm{h}} \right) - \frac{\psi^2}{2\mu_0 \rho^2 c_\mathrm{h}^2} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right] \rho \mathrm{d}V = 0$$

Fully Conservative Divergence Cleaning

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\frac{\boldsymbol{B}}{\mu_0 \rho} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_\psi + \frac{\psi}{\mu_0 \rho c_\mathrm{h}^2} \frac{\mathrm{d}\psi}{\mathrm{d}t} - \frac{\psi^2}{2\mu_0 \rho^2 c_\mathrm{h}^2} \frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{\psi^2}{\mu_0 \rho c_\mathrm{h}^3} \frac{\mathrm{d}c_\mathrm{h}}{\mathrm{d}t} \right] \rho \mathrm{d}V = 0$$

- Add $\nabla \cdot \boldsymbol{v}$ term to $\mathrm{d}\psi/\mathrm{d}t$ to account for compression/rarefaction
- Evolve ψ/c_h instead of ψ to account for time-varying c_h

Special thanks to Gábor Tóth for inspiration on this!

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\frac{\boldsymbol{B}}{\mu_0 \rho} \cdot \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \right)_\psi + \frac{\psi}{\mu_0 \rho c_\mathrm{h}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi}{c_\mathrm{h}} \right) - \frac{\psi^2}{2\mu_0 \rho^2 c_\mathrm{h}^2} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right] \rho \mathrm{d}V = 0$$

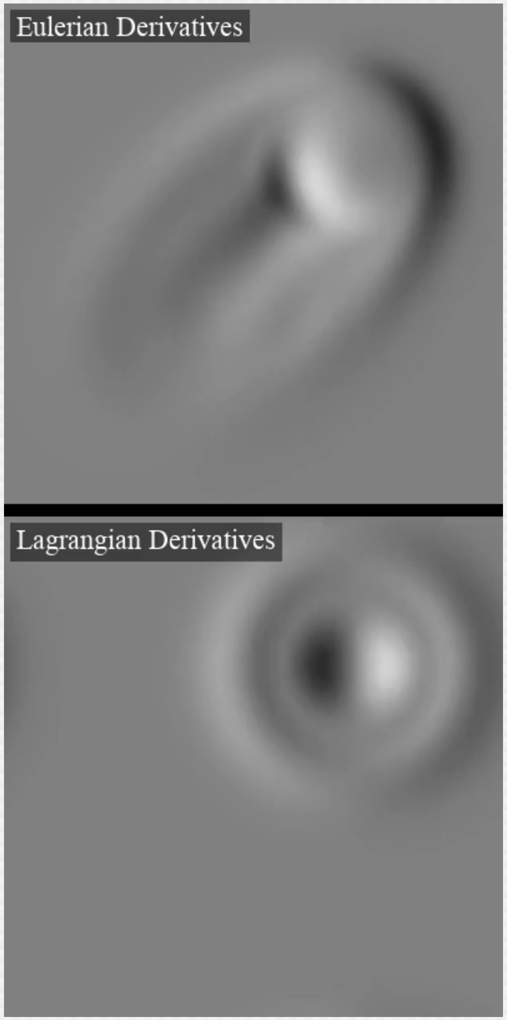
- New evolution equations (*not SPH specific!*)

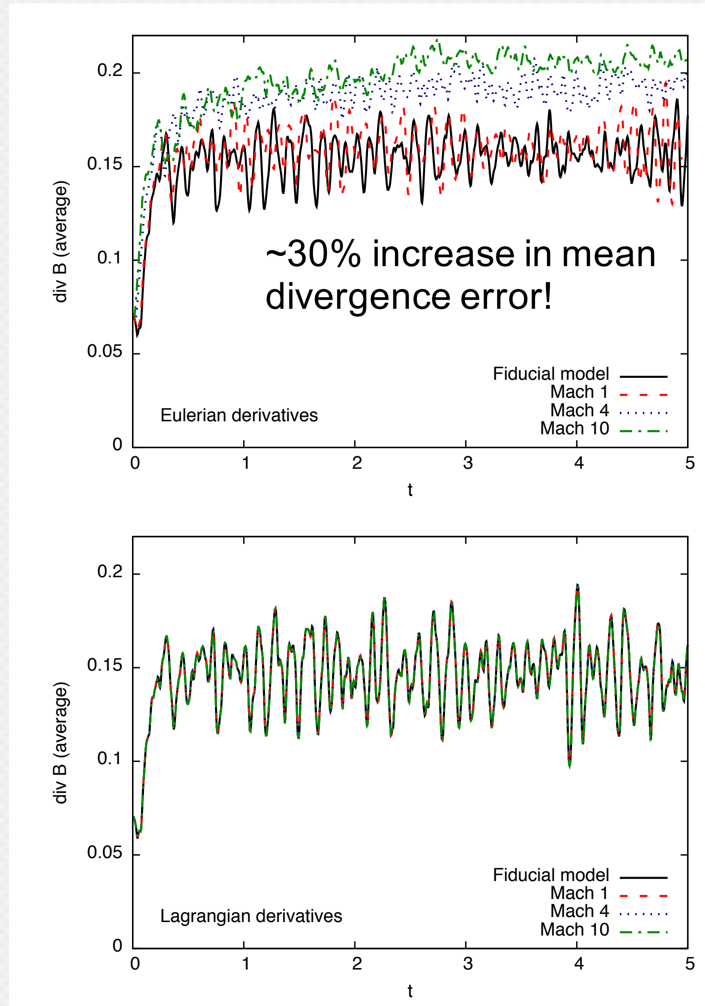
$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = -\nabla \psi \qquad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi}{c_\mathrm{h}} \right) = -c_\mathrm{h} \nabla \cdot \boldsymbol{B} - \frac{1}{\tau} \left(\frac{\psi}{c_\mathrm{h}} \right) - \frac{1}{2} \left(\frac{\psi}{c_\mathrm{h}} \right) \nabla \cdot \boldsymbol{v}$$

- Divergence waves move in Lagrangian frame, accounts for ρ, c_h, τ changes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{\sqrt{\rho} c_\mathrm{h}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi}{\sqrt{\rho} c_\mathrm{h}} \right) \right] - \frac{\nabla^2 \psi}{\rho} + \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{\sqrt{\rho} c_\mathrm{h}} \left(\frac{\psi}{\sqrt{\rho} c_\mathrm{h} \tau} \right) \right] = 0$$

Eulerian vs Lagrangian Derivatives





Tricco & Price (2012); Tricco, Price & Bate (2016)

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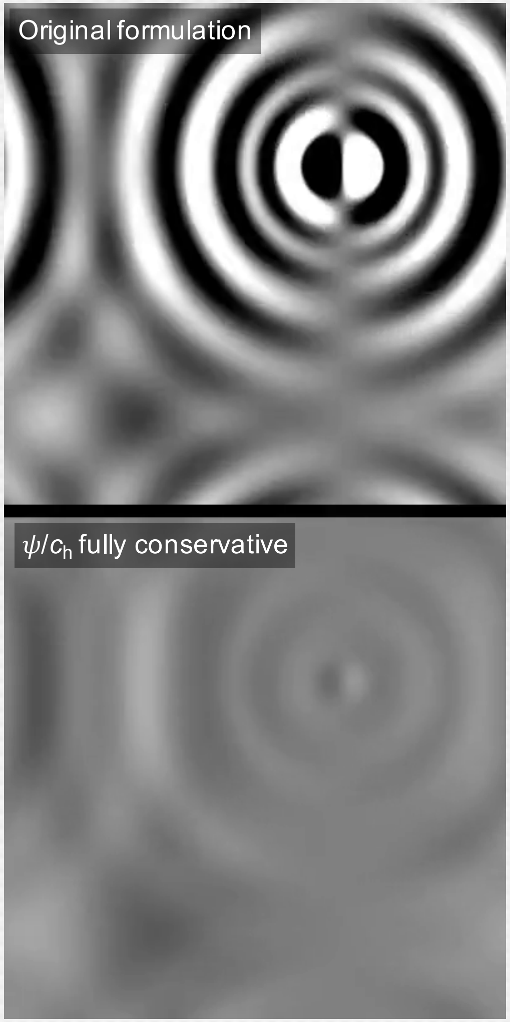
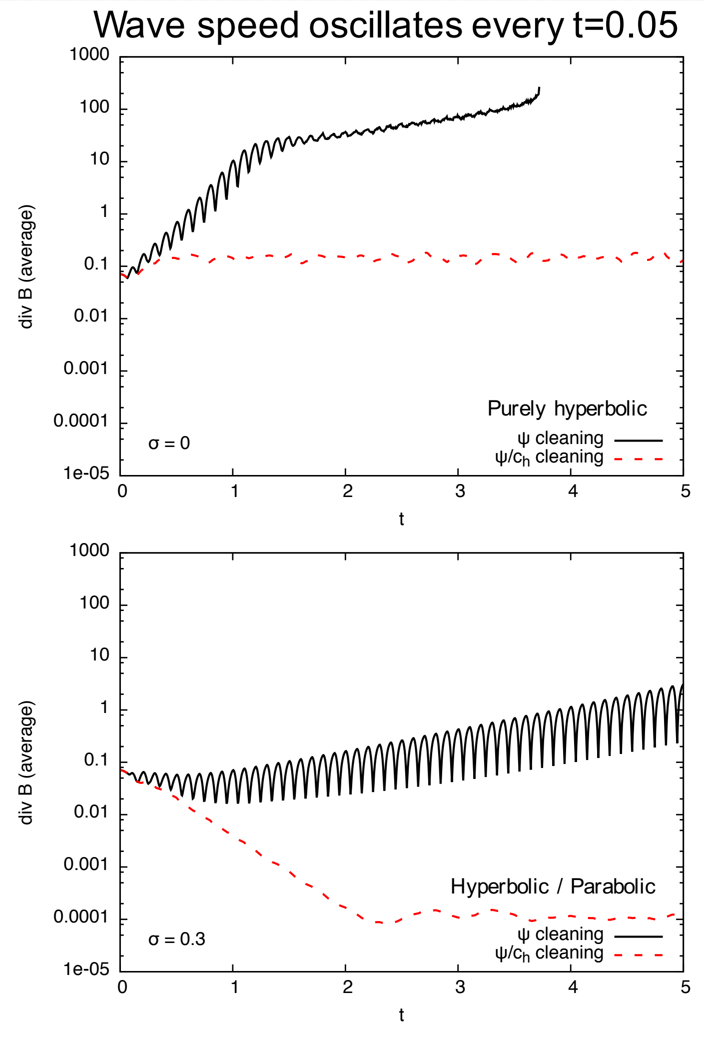
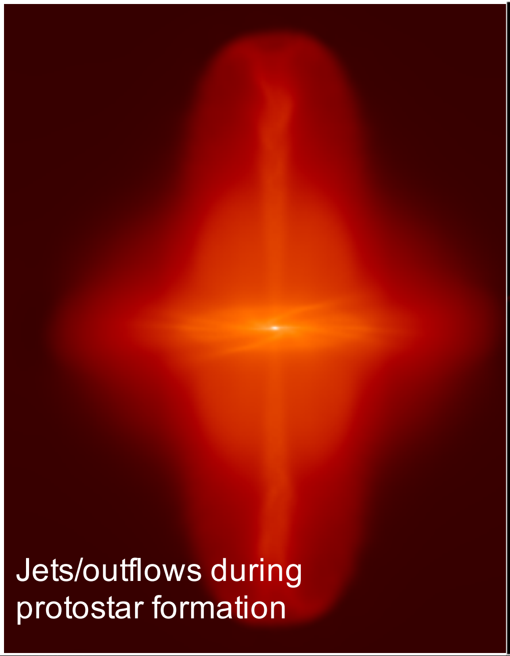
Discontinuous Time Variations of c_h

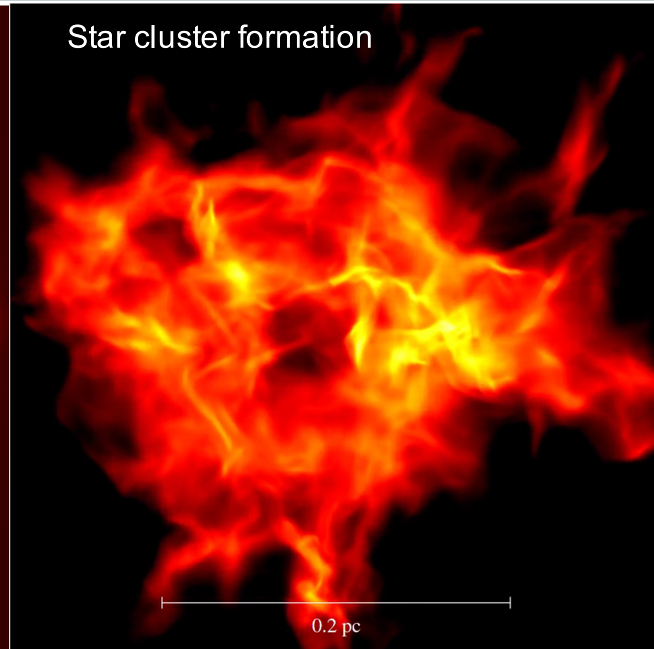




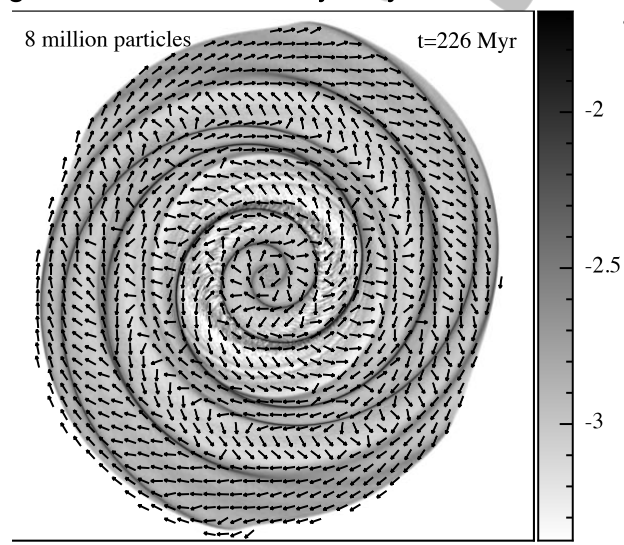
Phantom

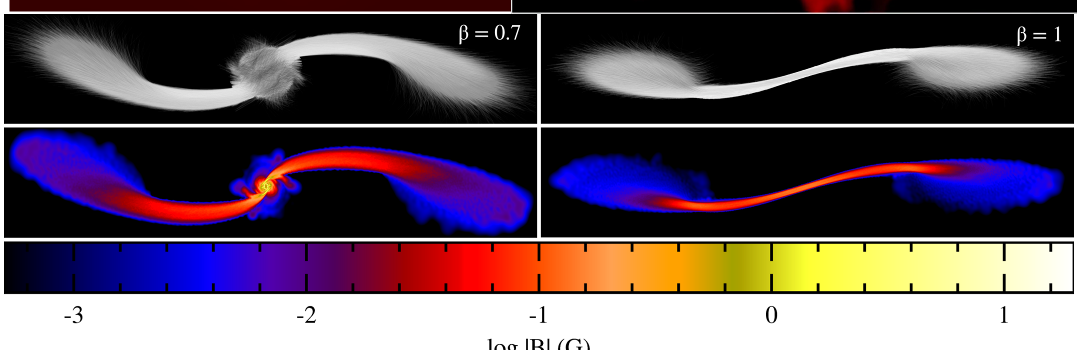
phantomsph.bitbucket.io



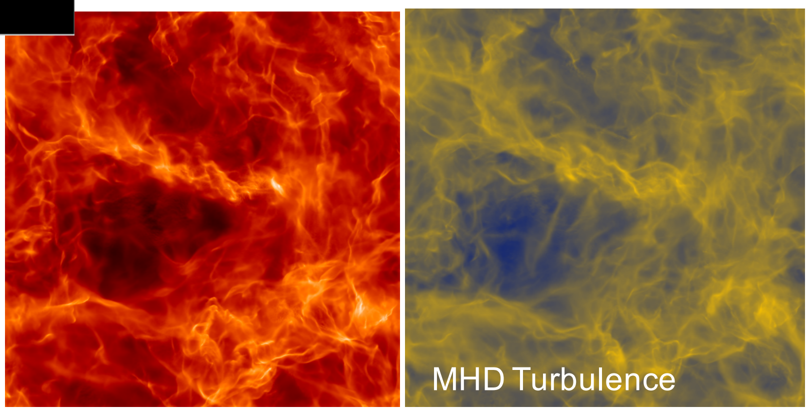


Magnetic field of the Milky Way





Magnetic fields in tidal disruption events



Price, Wurster, Nixon, Tricco et al (PASA submitted; arXiv:1702.03930)

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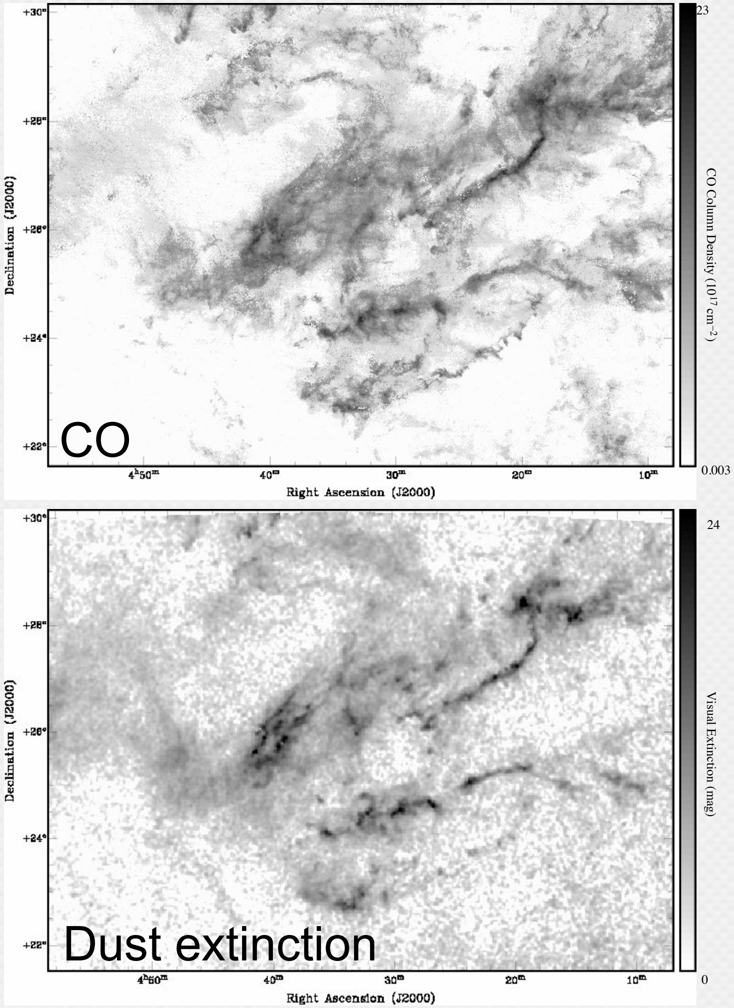
Credit Bruno Gilli/ESO

Dust in Molecular Clouds

- Molecular hydrogen emits weakly, so bulk of gas is effectively invisible
- 1% dust-to-gas mass ratio used to infer gas mass in dense, cold molecular clouds
- Turn to tracers: other molecules (CO, N_2H+), and/or dust
- Dust seems to correlate well with gas



Pineda et al (2010)

Is the Dust-to-Gas Ratio Constant?

But is the dust-to-gas ratio in molecular clouds uniformly 1%?

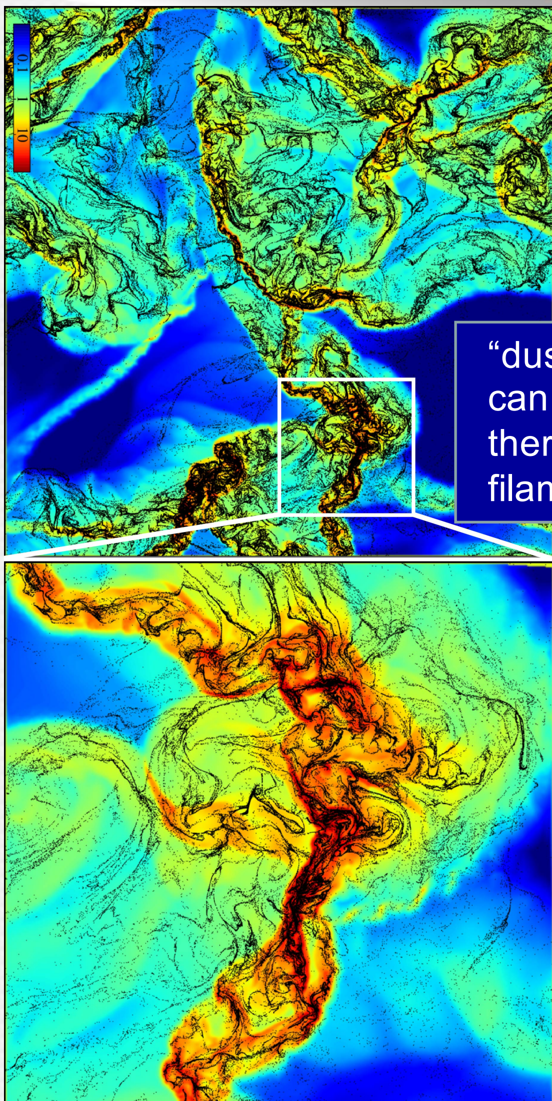
- **Anomalous extinction laws:**
  - $R_V = A_V / (A_B - A_V) = 3.1$ in diffuse clouds (fairly "universal"), but $R_V \sim 5$ in molecular clouds (Cardelli+ 1989; Weingartner & Draine 2001)
  - Implies molecular clouds have a different grain size distribution
- **Coreshine phenomenon:**
  - higher abundance of large grains (>1 micron) in dense filaments (Pagani+ 2010; Steinacker+ 2010; Lefevre+ 2014)
- **Ophiuchus A:**
  - Spatial mismatch between gas and dust (Di Francesco+ 2004; Friesen+ 2014)
  - mean 1.1% dust-to-gas ratio, but with local fluctuations of 0.5% to 10% (Liseau+ 2015)

Dust is not Gas!

- Dust behaves dynamically different than gas
- Could supersonic turbulence cause dynamical variations in the dust-to-gas ratio?

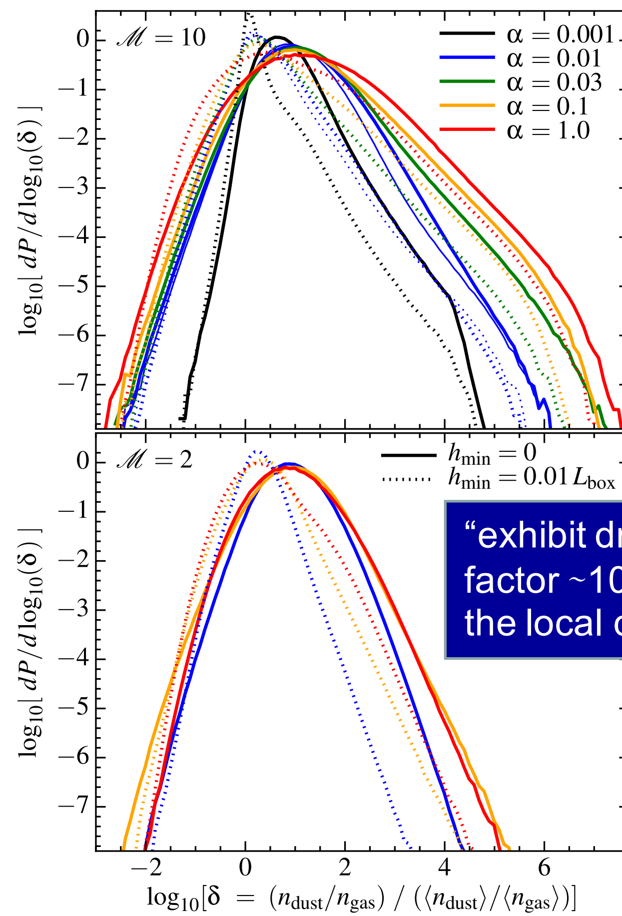
- Analytic estimates for 0.1 micron dust grains in molecular clouds
  - density $\rho \sim 10^{-20}$ g/cm^3, sound speed ~ 0.2 km/s,
  - grain size $s_{\rm grain} \sim 0.1$ micron, intrinsic grain density $\rho_{\rm grain} \sim 3$ g/cm^3
  - Drag stopping timescale $t_s \sim \rho_{\rm grain}\, s_{\rm grain} / \rho \sim 10^3$ yrs << 10^6 dynamical time
  - Expect well-coupled mixture of dust and gas. High drag regime.
- But this assumes cloud is homogenous **and *ignores turbulence***!

Simulating Dust in Turbulent Clouds



“dust filaments can exist where there is no gas filament at all"

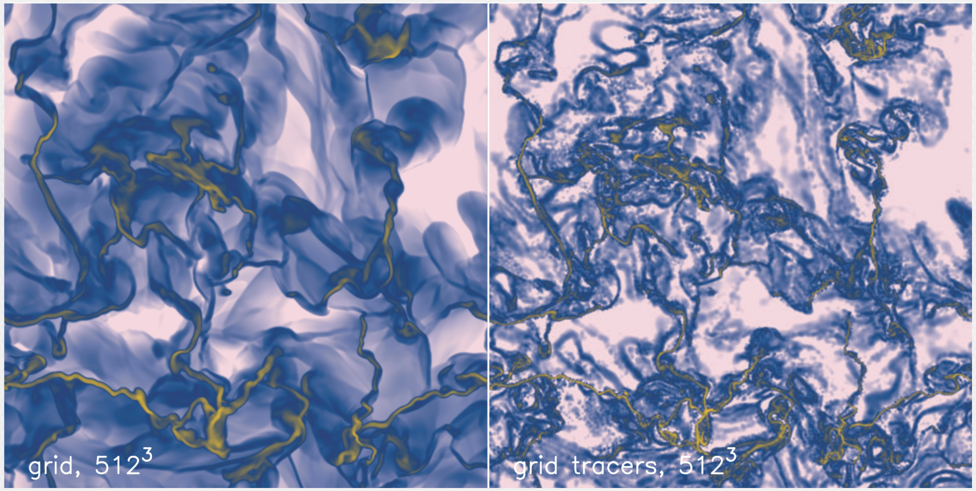
Hopkins & Lee (2016); Lee, Hopkins & Squire (2017)



“exhibit dramatic (exceeding factor ~1000) fluctuations in the local dust-to-gas ratio"

Two-Fluid Dust/Gas at High Drag

- Two fluid dust/gas must *resolve 'stopping length'* of dust grains *in the gas*; $l_s \sim c_s * t_s$ (Price & Laibe 2012)
  - For ~0.1 micron dust grains in cold, dense molecular clouds, require ~1600^3 gas elements (even stricter in dense filaments!)
- Tracer particles in compressible turbulence are known to suffer from numerical artefacts (Price & Federrath 2010; Genel et al 2013)



"it is notable that a dense shock structure appears in the [tracer particles] that is completely absent from both SPH and grid density fields"

"the resulting [density] PDFs show a strong deviation from a lognormal distribution, particularly in the high density tail"

"the velocity field tracers display structures that do not exist in the gas distribution"

One-Fluid: Accurate for High Drag

One fluid approach (Laibe & Price 2014a,b; Price & Laibe 2015)

- Change of variables; each element is mixture of dust and gas

$$\frac{\partial \rho_g}{\partial t} + \nabla \cdot (\rho_g \boldsymbol{v}_g) = 0$$
$$\frac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \boldsymbol{v}_d) = 0$$
$$\rho_g \left(\frac{\partial \boldsymbol{v}_g}{\partial t} + \boldsymbol{v}_g \cdot \nabla \boldsymbol{v}_g \right) = \rho_g \boldsymbol{f} + K(\boldsymbol{v}_d - \boldsymbol{v}_g) - \nabla P_g$$
$$\rho_d \left(\frac{\partial \boldsymbol{v}_d}{\partial t} + \boldsymbol{v}_d \cdot \nabla \boldsymbol{v}_d \right) = \rho_d \boldsymbol{f} - K(\boldsymbol{v}_d - \boldsymbol{v}_g)$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho(\nabla \cdot \boldsymbol{v})$$
$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \boldsymbol{f}$$
$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{1}{\rho}\nabla \cdot (\epsilon t_s \nabla P)$$

Terminal velocity approximation

- Works well when dust and gas well coupled
- No need to worry about drag spatial resolution criterion
- Accurate while $\epsilon t_s < \Delta t_{cour}$; becomes easier to satisfy time criterion in dense filaments where t_s decreases

Supersonic Dusty Turbulence

Modelled turbulent dynamics of dusty molecular clouds using the SPH code Phantom (Price et al, 2017; phantomsph.bitbucket.io)

- Not concerned with grain growth or destruction
- No self-gravity (not trying to make stars) or magnetic fields

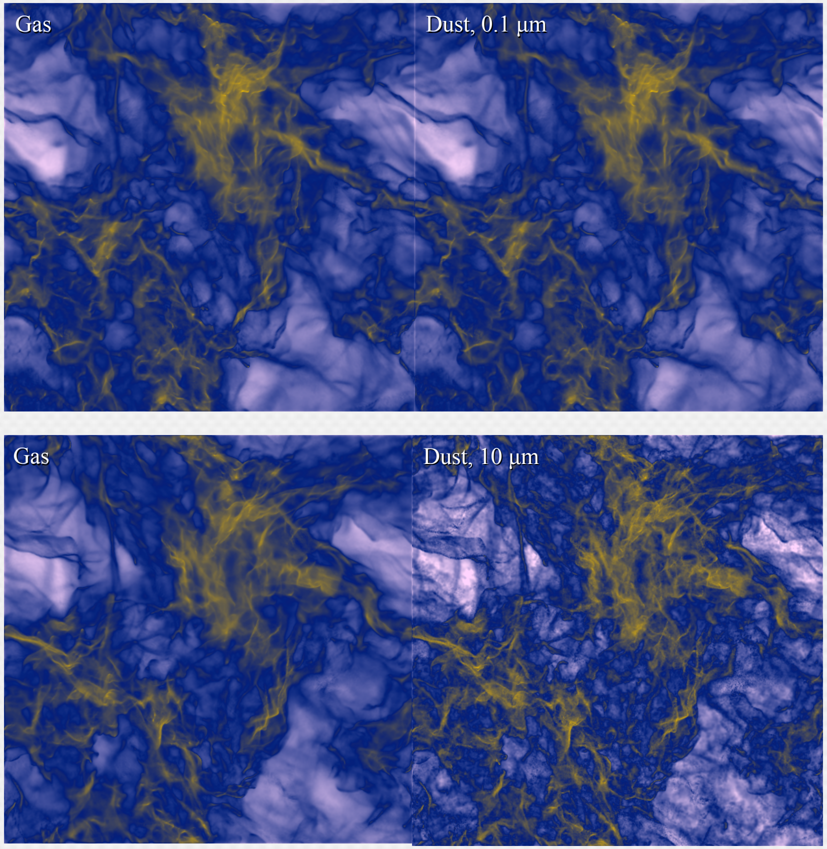
Initial Conditions:

- L = 3 parsec, Density = 10^{-20} g/cm^3
- Isothermal gas with T= 11.5 K, c_s = 0.2 km/s
- Mach 10 turbulence driven on large scales for ~14 Myr

Dust Physics:

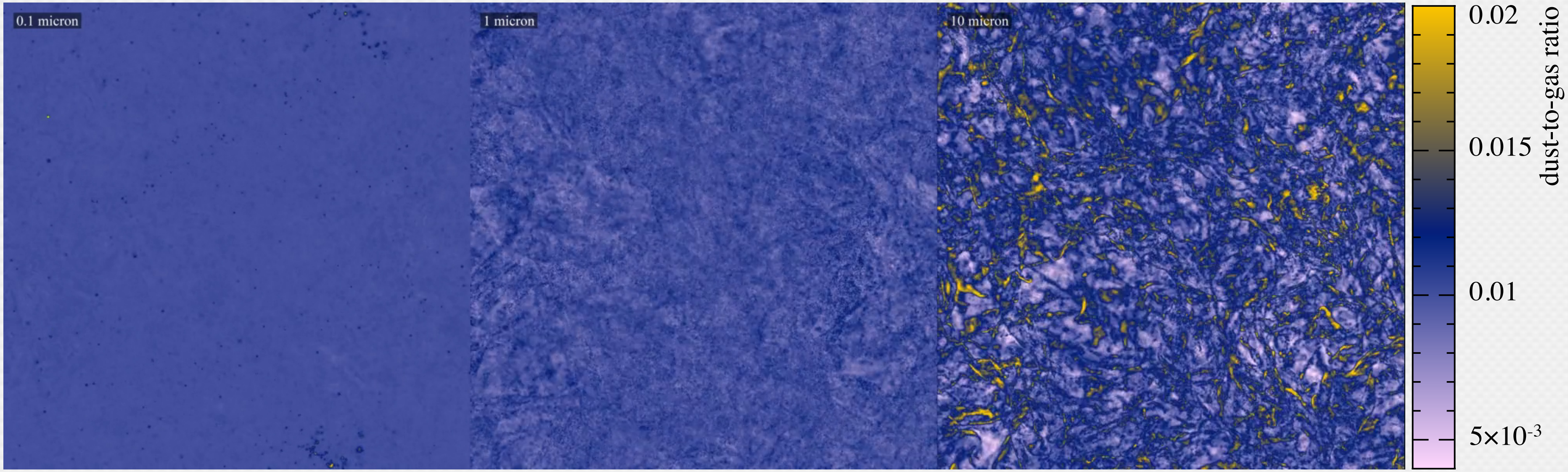
- 0.1, 1 and 10 micron dust grains (3 separate simulations)
- Initially uniform 1% dust-to-gas mass ratio
- One fluid dust/gas solver

Column Dust and Gas Density



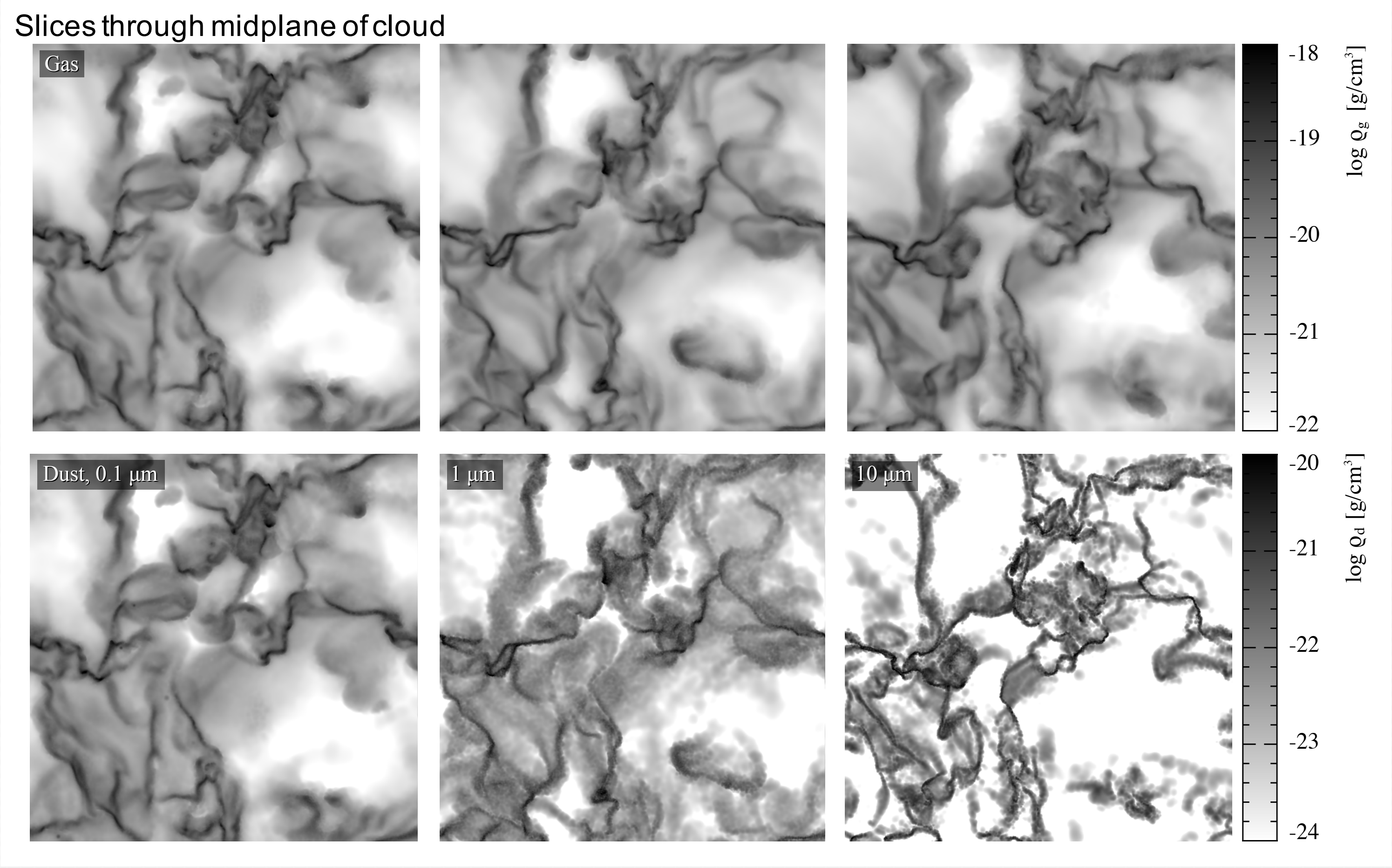
- Large-scale column dust density traces gas column density for 0.1 – 10 microns
- Some variation in dust column density relative to gas column density for 10 micron

Column Dust-to-Gas Ratio



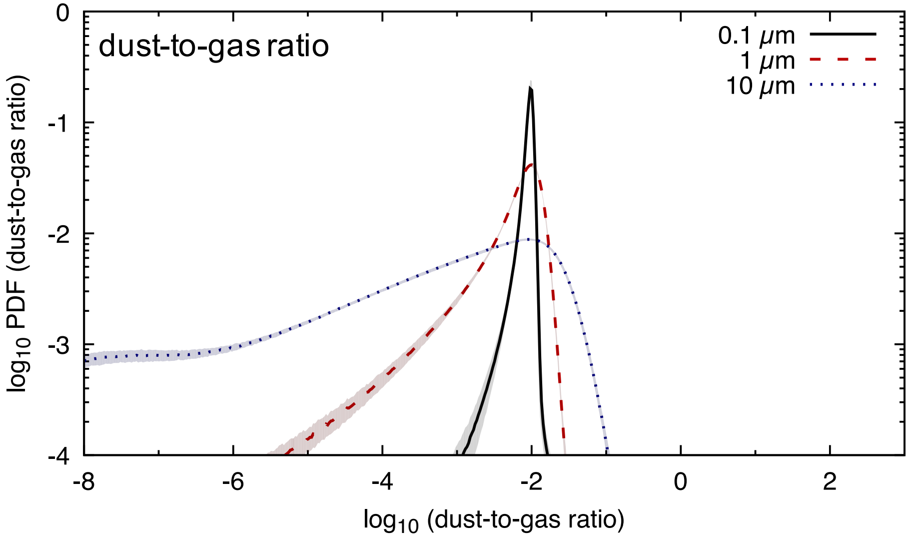
- Almost no variation in dust-to-gas ratio for 0.1 micron grains
- Large, 10 micron grains typical variations of ~2-3x (max ~10x)

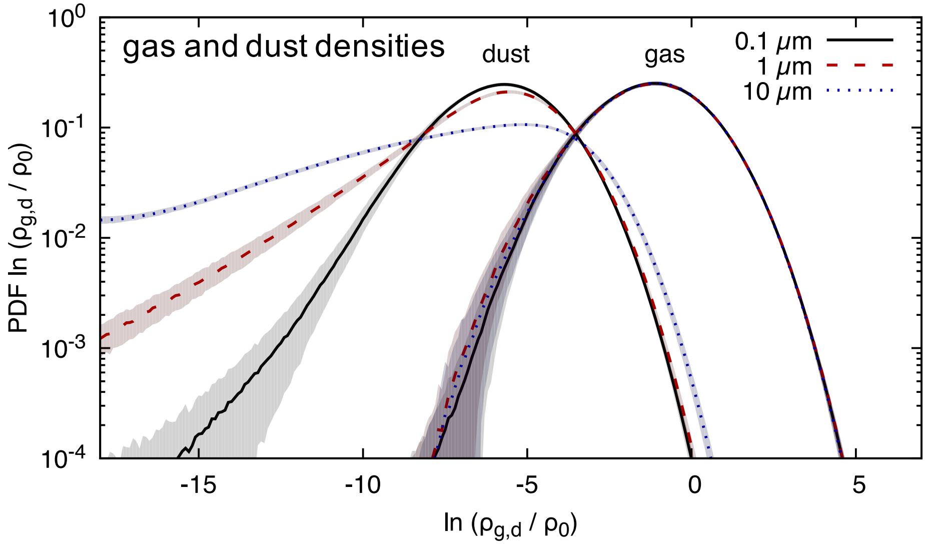
‘Size-sorting’ of Dust



- 0.1 micron grains remain well-coupled to the gas throughout the cloud
- Large grains preferentially concentrated in filaments
- Up to 10x increase of 10 micron grains in filaments

Probability Distributions

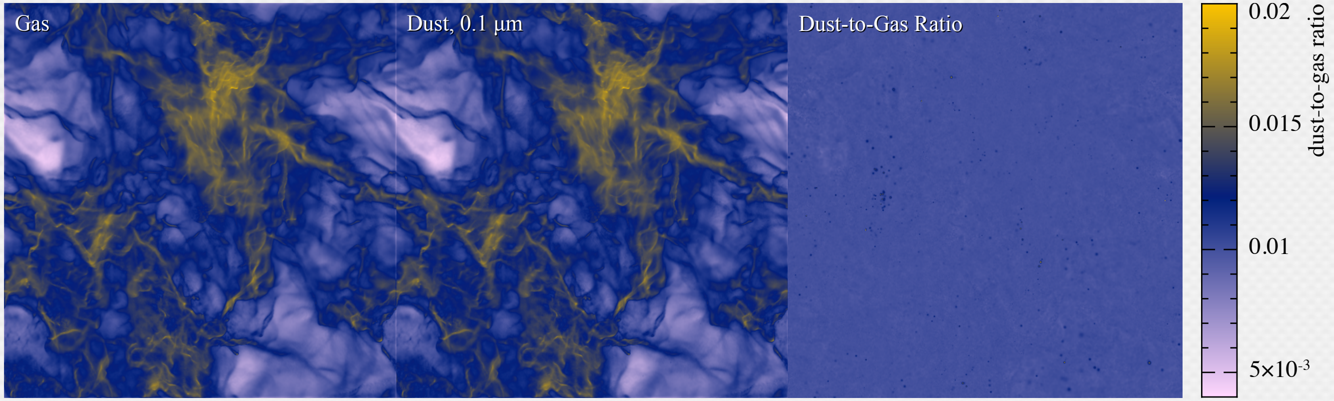




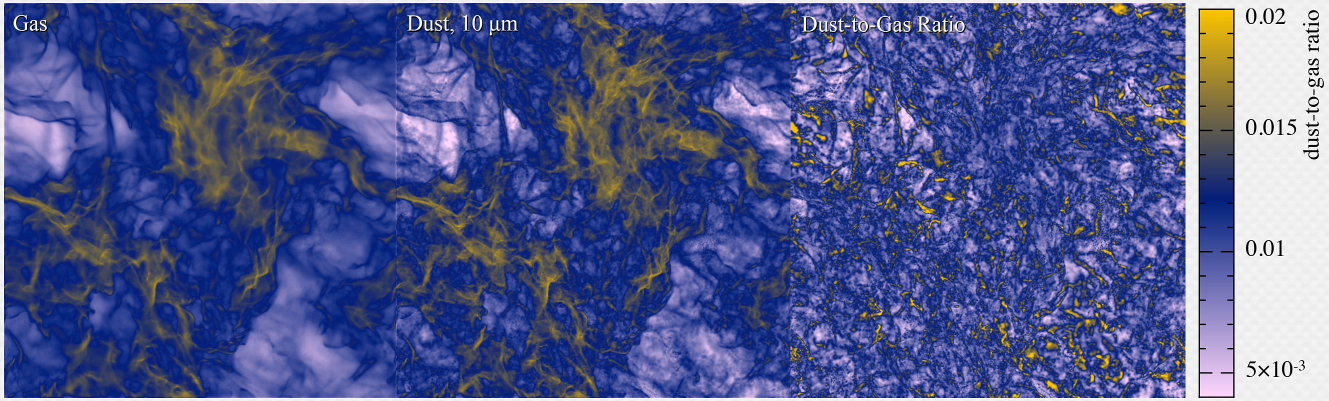
- ~0.1 micron grains:
  - Sharply peaked PDF of dust-to-gas ratios at 1%
  - Dust density distribution matches gas (well-coupled throughout cloud)
- 1-10 micron grains:
  - PDFs broaden with increasing grain size due to 'size-sorting'
  - Turbulence causes dynamical transfer of dust mass into high density filaments

Is the dust-to-gas ratio constant?

- **Yes!** For ~0.1 micron grains, turbulence almost no effect since dust is well-coupled to gas throughout the cloud



- **No!** For ≥ 1 micron grains, turbulence causes typical variations ~2-3x



Summary

- New formulation of “constrained” hyperbolic divergence cleaning accounts for local variations in the wave cleaning speed (Tricco, Price & Bate, 2016):
  - Now fully conservative!
  - New divergence cleaning equations not SPH specific!
  - Has been used to do practical simulations (with Phantom SPH code): MHD turbulence, magnetic fields in galaxies, protostar formation, star cluster formation, self-gravitating accretion discs, tidal disruption events
- Dynamical effect of supersonic turbulence on dust in molecular clouds (Tricco, Price & Laibe, 2017):
  - Contrary to Hopkins et al, do not find orders of magnitude fluctuations for ~0.1 micron dust grains
  - Local variations of dust-to-gas ratio for large grains (>1 micron) typically of ~40%, up to 10x for large grains
  - ‘size-sorting’: concentration of large grains into filaments due to changing dust-stopping times throughout turbulent clouds