Hyperbolic Divergence Cleaning for Smoothed Particle Magnetohydrodynamics

New Advances in SPMHD

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Magnetic Fields in Astrophysics

- Magnetic fields are important for many astrophysical problems.
- SPH widely used to model gas dynamics in astrophysics:
  - simple,
  - inherently adaptive,
  - strong conservation of physical quantities.

SPMHD

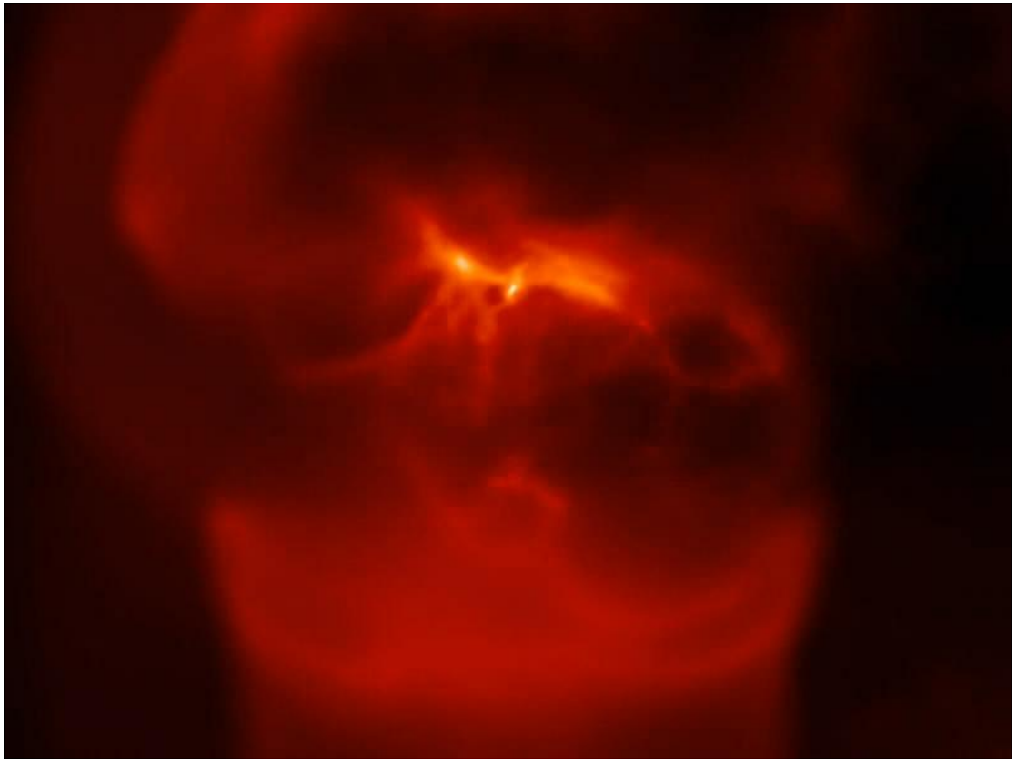
Most important criterion: $\nabla \cdot \mathbf{B} = 0$

- Momentum equation:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{J} \times \mathbf{B} + \mathbf{B}(\nabla \cdot \mathbf{B})$$

- Lorentz force + extra divergence term.
- In SPMHD, subtract out extra divergence term for numerical stability, but lose momentum conservation!

What happens if $\nabla \cdot B$ is large?



- Collapse of magnetised molecular cloud core (edge on view).
- Strong central divergence causes the protostar (and most everything else) to be ejected out of the disc.

Approaches to Control Divergence

1) Strong artificial resistivity:

$$\left(\frac{\partial \mathbf{B}}{\partial t}\right)_{\rm resist} \equiv \eta \nabla^2 \mathbf{B} = \eta \nabla \left(\nabla \cdot \mathbf{B}\right) - \eta \nabla \times \left(\nabla \times \mathbf{B}\right)$$

- Some measure of divergence control for free,
- But weakens physical field as well.

2) Euler Potentials:

$$\mathbf{B} = \nabla \alpha \times \nabla \beta$$

- Can't represent certain field configurations.

Hyperbolic Divergence Cleaning

Couple scalar field ψ to magnetic field:

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\psi} = -\nabla\psi$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c_h^2 \nabla \cdot \mathbf{B} - \frac{\psi}{\tau}$$

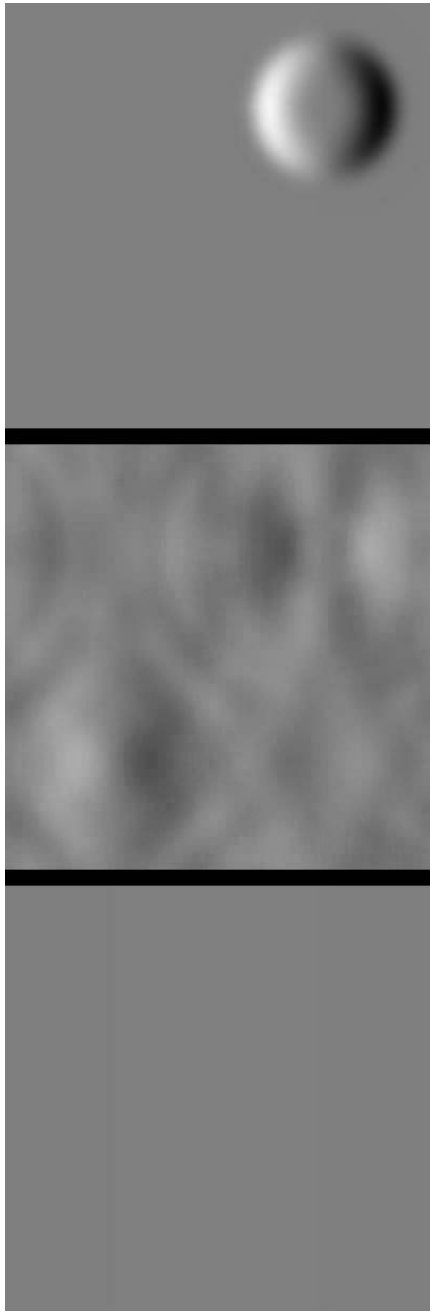
Produces damped “divergence” waves:

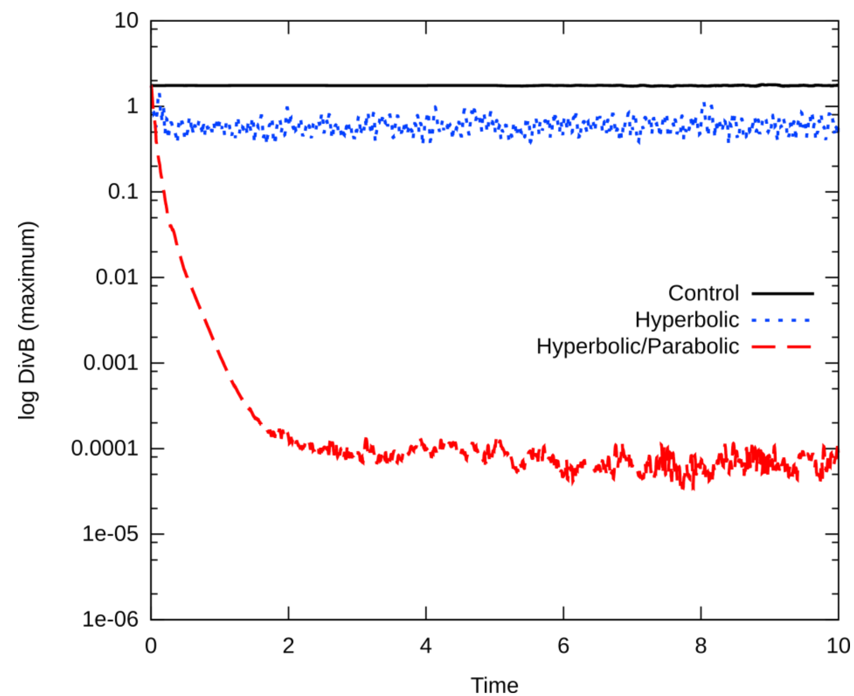
$$\frac{\partial^2(\nabla \cdot \mathbf{B})}{\partial t^2} - c_h^2 \nabla^2(\nabla \cdot \mathbf{B}) + \frac{1}{\tau}\frac{\partial(\nabla \cdot \mathbf{B})}{\partial t} = 0$$

No Cleaning

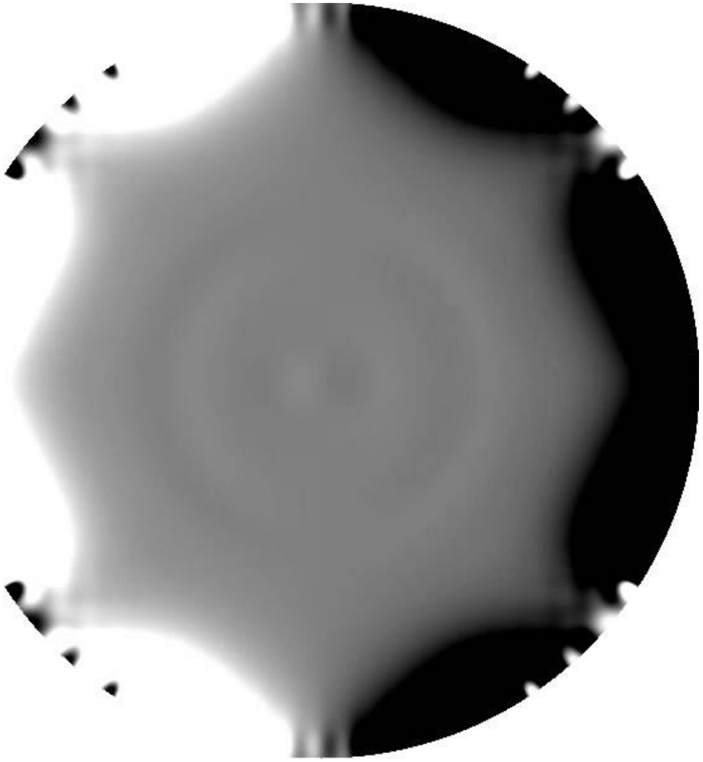
Undamped Cleaning

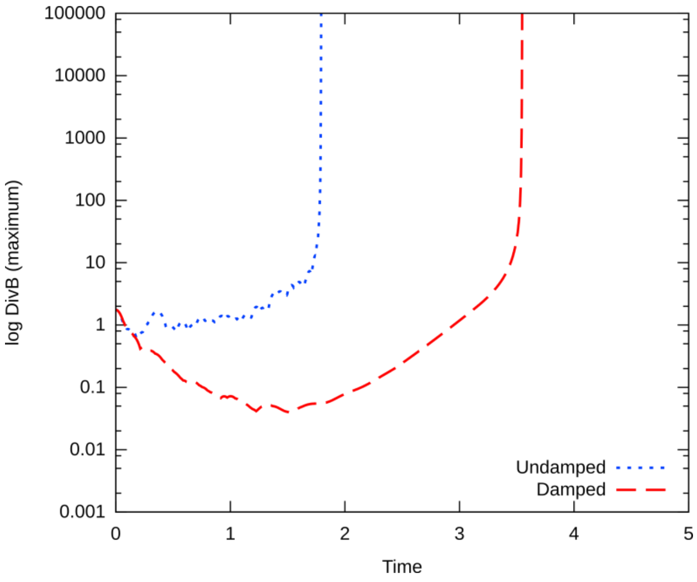
Damped Cleaning



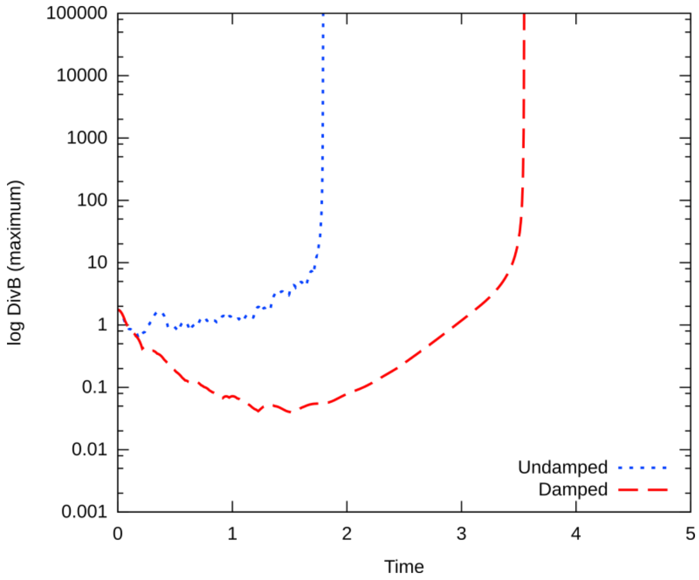


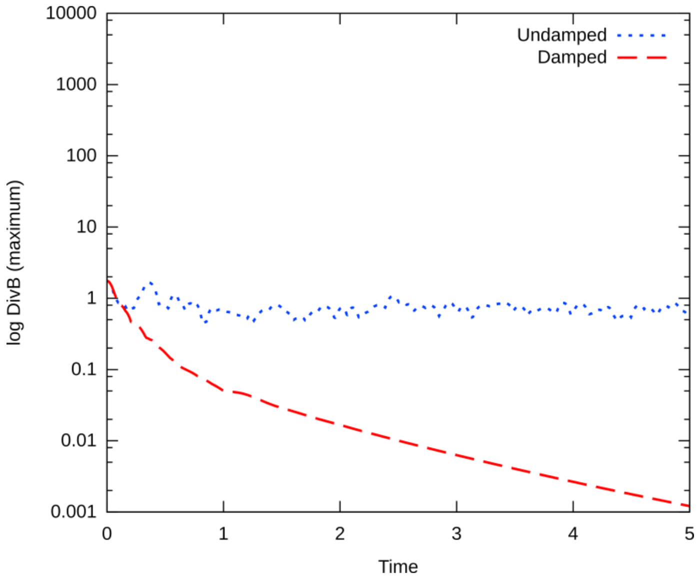
Stability Improvement





Stability Improvement





Stable Formulation

Define energy of ψ field:

$$e_\psi \equiv \frac{\psi^2}{\mu_0 \rho c_h^2}$$

Use as part of system Lagrangian:

$$L = \int \left(\tfrac{1}{2}\rho \mathbf{v}^2 - \rho u - \frac{\mathbf{B}^2}{2\mu_0} - \frac{\psi^2}{2\mu_0 c_h^2} \right) \mathrm{d}V$$

New evolution equation:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c_h^2 \nabla \cdot \mathbf{B} - \frac{\psi}{\tau} - \tfrac{1}{2}\psi \nabla \cdot \mathbf{v}$$

SPMHD Implementation

Two standard measures of first derivative:

$$(\nabla \cdot \mathbf{B})_a = \rho_a \sum_b m_b \left[\frac{\mathbf{B}_a}{\Omega_a \rho_a^2} \cdot \nabla_a W_{ab}(h_a) + \frac{\mathbf{B}_b}{\Omega_b \rho_b^2} \cdot \nabla_a W_{ab}(h_b) \right]$$ (Symmetric)

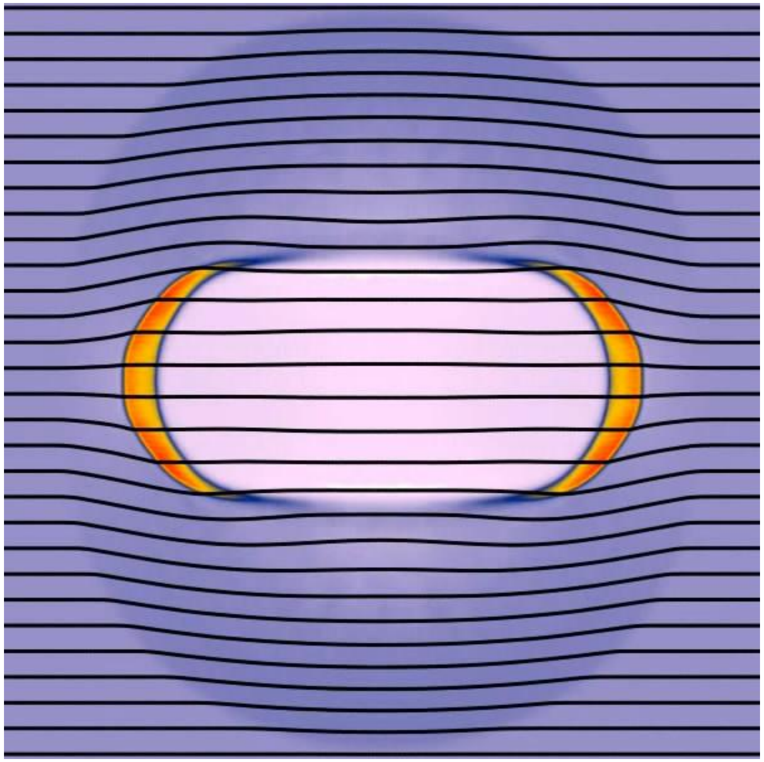
- Irregularities in magnetic field + Particle disorder

$$(\nabla \cdot \mathbf{B})_a = -\frac{1}{\Omega_a \rho_a} \sum_b m_b \left(\mathbf{B}_a - \mathbf{B}_b \right) \cdot \nabla_a W_{ab}(h_a)$$ (Difference)

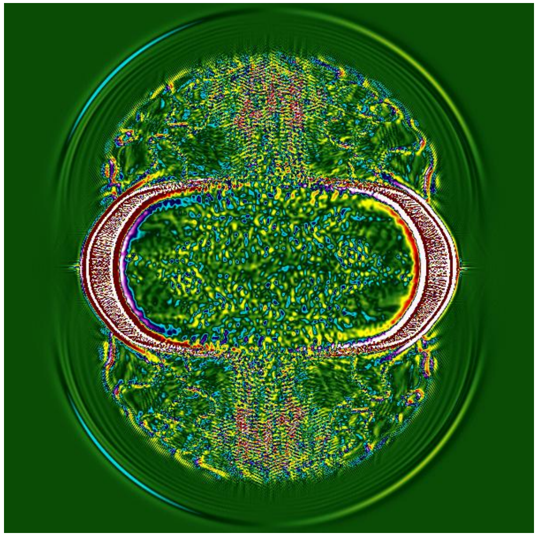
- Irregularities in magnetic field

Which of these to use?

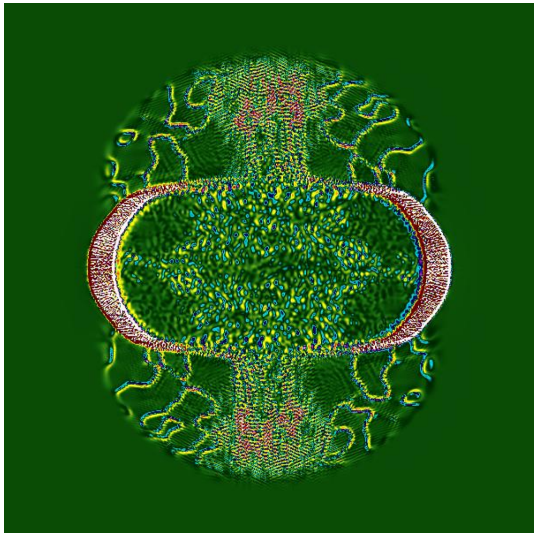
SPMHD Implementation



Density of 2D MHD Blast wave.



Symmetric $\nabla \cdot B$

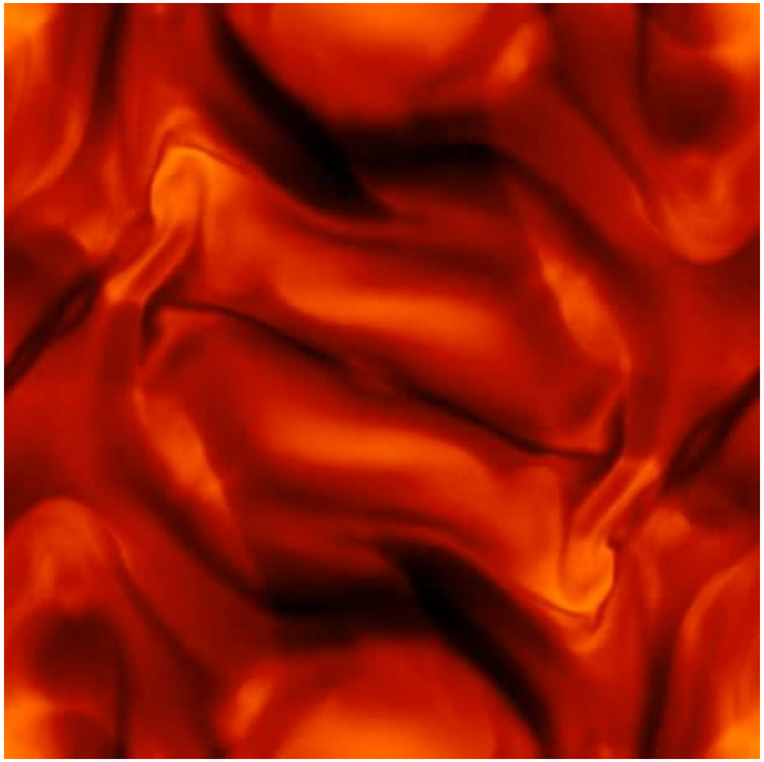


Difference $\nabla \cdot B$

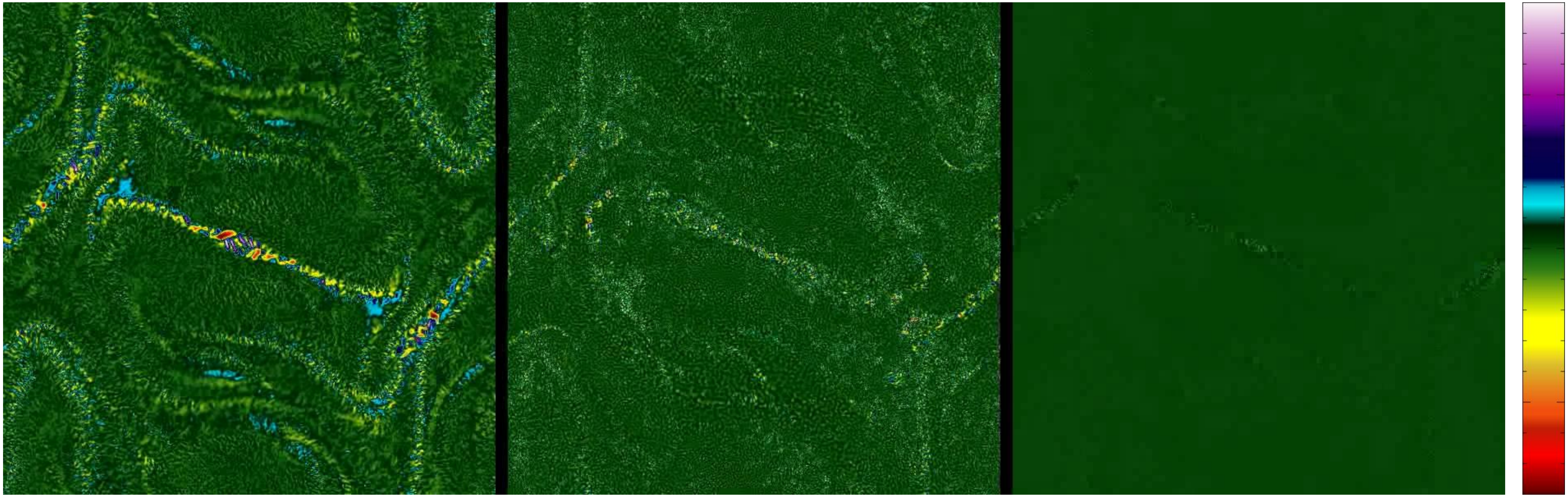
Results in significant magnetic energy loss.

Orszag-Tang Vortex

- Multiple interacting classes of shocks.
- Compare results for
  - resistivity,
  - Euler potentials,
  - divergence cleaning.

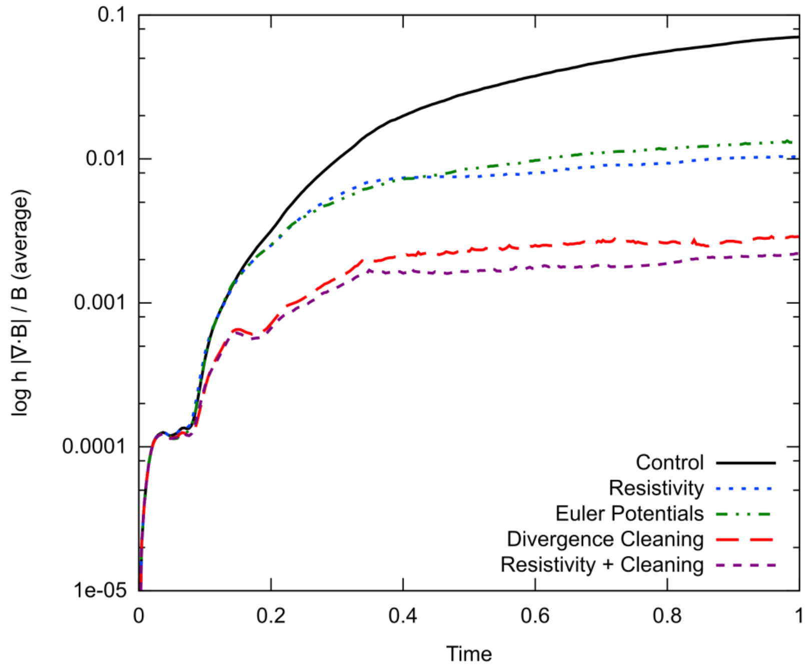


Orszag-Tang: $\nabla \cdot B$



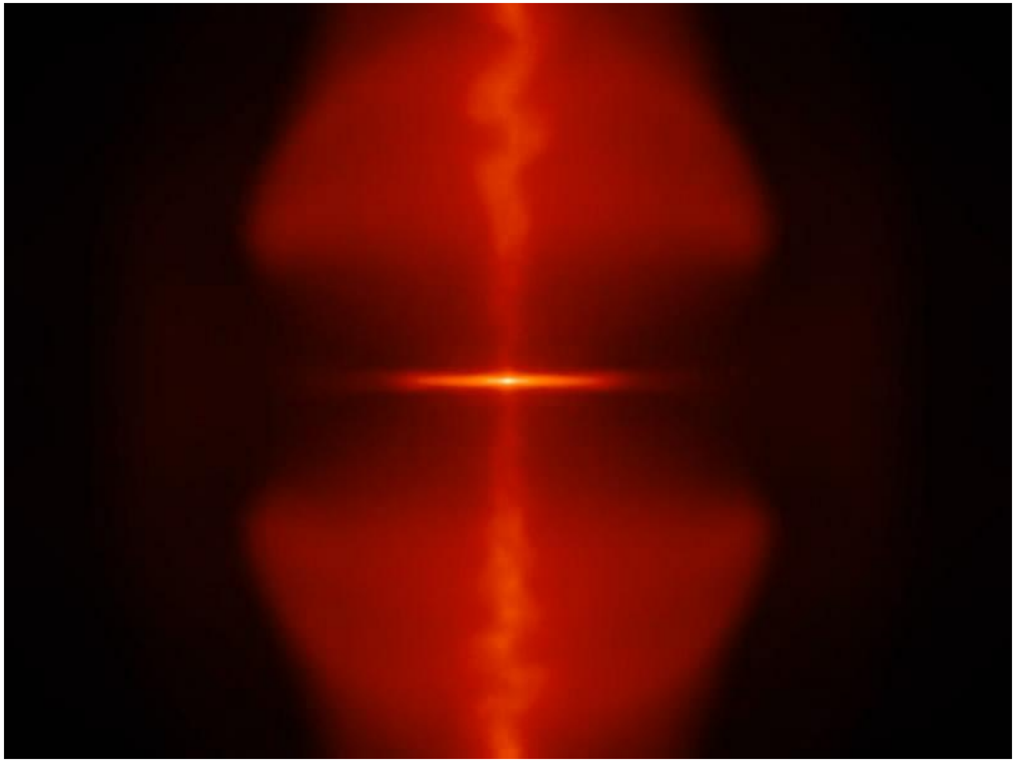
Resistivity Euler Potentials Divergence Cleaning

Orszag-Tang: Comparisons



Average divergence in the Orszag-Tang vortex. Divergence cleaning provides an order of magnitude improvement over resistivity or Euler potentials.

Collapse with Divergence Cleaning



- With divergence cleaning, collapse remains stable!
- Momentum conservation improved by 2 orders of magnitude.

Conclusions

- Stable implementation of hyperbolic divergence cleaning for SPMHD.
- Unlike resistivity or Euler potentials, no drawbacks!
- and ~Order of magnitude reduction in average divergence over existing methods.
- For more details:
  - Tricco, Price, J. Comp. Phys (2012) submitted.