Computer Science 6789 (Winter 2014):
Assignment #1
Due: 10:30 AM on Thursday, January 30, 2014

1. **(25 marks)** For each of the algorithms below, derive an **asymptotic worst-case**, *i.e.*, Big-Oh, complexity function $O(f(n))$. Briefly explain the reasoning behind each derivation.

   (a) **(5 marks)**

```
sum = 13
cond = false
for i = 1 to n do
    for j = 1 to 7 do
        sum = sum / (i + j)
    if COND(sum)
        cond = true
    else
        for j = 1 to n do
            sum = sum - j
if cond
    for j = 1 to n * log(n) do
        sum = sum + (i/j)
```

   (b) **(5 marks)**

```
sum = 42
for i = 1 to n * n do
    j = 1
    finished = true
    while ((j <= n) and (not finished)) do
        for k = 1 to log(n) * n do
            sum = sum / (k * i) + j
        if COND(sum)
            finished = true
```

   Assume that method COND() runs in $(n + 13)$ timesteps.

   (c) **(5 marks)**

```
sum = 42
for i = 1 to n * n do
    j = 1
    finished = false
    while ((i <= n) and (not finished)) do
        for k = 1 to log(n) * n do
            finished = true
        if COND(sum)
            sum = sum / (k * i) + j
```

Assume that method COND() runs in $(n + 13)$ timesteps.

(d) **(5 marks)**

```
sum = 42
for i = 1 to n * log(n) do
    j = 1
    finished = false
    for k = 1 to  n do
        if COND(sum)
            sum = sum / (k * i) + j
        while ((j <= n) and (not finished)) do
            finished = true
```

Assume that method COND() runs in $(n + 13)$ timesteps.

(e) **(5 marks)**

```
n = number of elements in set U
for each binary vector v of length n do
    if the number of 1's in v is odd then
        add v to U
    for each subset s of the indices of elements in v with value 0 do
        if the sum of the elements in s is odd then
            add s to U
```

Assume that adding an element to a set takes constant, *i.e.*, 0(1), time.

2. **(20 marks)** Prove or disprove the following:

   (a) **(5 marks)** $f(n) = n^d + 10n^2$, where d is some integer constant greater than or equal to 2, is $O(n^d)$.

   (b) **(5 marks)** $f(n) = 1000 \times 2^n$ is $0(2^{\frac{n}{2}})$.

   (c) **(5 marks)** $f(n) = 10^{1000} \times 3^n$ is $0(2^n)$.

   (d) **(5 marks)** $f(n) = 2^n$ is $0(n^c)$, where c is some integer constant greater than or equal to 100..

3. **(30 marks)** Consider the following decision problems:

   DUMBBELL SUBGRAPH (DS)
   Input: An undirected graph $G = (V, E)$ and two positive integers $k, l \geq 1$.
   Question: Are there two cliques C_1 and C_2 and a simple path P in G such that C_1 and C_2 have $\geq k$ vertices apiece, P has $\geq l$ edges, P connects C_1 and C_2, the cliques and path do not have any edges in common, and the only vertices that P shares with C_1 (C_2) is its connection-vertex?

Bounded-Weight Subset Cover (BWSSC)
Input: A set $I = \{i_1, \ldots, i_a\}$ of items, a set $R = \{r_1, \ldots, r_b\}$ of subsets of I, an integer-valued subset-weight function $w()$ such that for each $r_x \in R$, $w(r_x) > 0$, a subset $N \subseteq I$, and integers $0 < k_1 \leq k_2$.
Question: Is there is a subset $R' \subseteq R$ such that $\cup_{r \in R'} r = N$ and $k_1 \leq \sum_{r \in R'} w(r) \leq k_2$?

a) (15 marks) Prove that problem DS is NP-complete by (1) showing that this problem is in NP and (2) giving a polynomial-time many-one reduction (algorithm + proof of correctness) to this problem from an NP-hard problem.

b) (15 marks) Prove that problem BWSSC is NP-complete by (1) showing that this problem is in NP and (2) giving a polynomial-time many-one reduction (algorithm + proof of correctness) to this problem from an NP-hard problem.

4. **(75 marks)** For each of the following problems, either prove that this problem is in P (by giving a polynomial-time algorithm) or prove that this problem is NP-complete (by showing that this problem is in NP and giving a polynomial-time many-one reduction (algorithm + two-way proof of correctness) to this problem from an NP-hard problem).

Hints:

* Two of the problems below are solvable in polynomial-time and three are NP-complete.
* When showing NP-hardness for the problems below, you may find it convenient to use reductions from some of the NP-complete problems given in the list in Lecture #2 of the course notes.

(a) **(15 marks)**

c-ZPath (cZP) (c is an integer constant of value ≥ 1)
Input: An undirected graph $G = (V, E)$.
Question: Can the vertices in G be colored with two colors such that (1) no edge's endpoint-vertices have the same color and (2) there is a path in this colored version of G with $\geq c$ edges in which no vertex or edge repeats and the vertex-colors alternate for the entire length of the path?

(b) **(15 marks)**

Bi-Robustness (BR)
Input: An undirected graph $G = (V, E)$, a pair of vertices $x, y \in V$, and a positive integer k.
Question: Are there two paths P_1 and P_2 linking x and y in G such that each path contains at least k edges and $P1$ and $P2$ do not intersect, *i.e.*, $P1$ and $P2$ do not have any edges in common, no vertex or edge occurs more than once in $P1$ or $P2$, and the only vertices $P1$ and $P2$ have in common are x and y?

(c) **(15 marks)**

c-ACCUMULABILITY (cA) (c is an integer constant of value ≥ 1)
Input: A directed graph $G = (V, A)$ a set of subsets $S = \{S_1, S_2, \ldots, S_c\}$ of V, a pair of vertices $x, y \in V - (\bigcup_{1=1}^{c} S_i)$, and a positive integer k.
Question: Is there a path in G from x to y such that this path includes at least one vertex from each of the subsets in S and the number of edges in this path is $\leq k$?

(d) **(15 marks)**

k-PCLIQUE (kPC)
Input: An undirected graph $G = (V, E)$ and a positive integer k.
Question: Is there a subset $V' \subseteq V$ such that $|V'| \geq k$ and there is a path in G between each pair of vertices in V'?

(e) **(15 marks)**

0/1 NESTED KNAPSACK (01NK)
Input: A set of items U, item size- and value-function $s()$ and $v()$, a positive integer k, and positive integers B_1 and B_2 such that $B_1 \leq B_2$.
Question: Are there subsets U_1 and U_2 of U such that $U_1 \subset U_2$, $\sum_{u \in U_1} s(u) \leq B_1$, $\sum_{u \in U_2} s(u) \leq B_2$, and $\sum_{u \in U_2} v(u) \geq k$?