Characterizing Finite Undirected Multigraphs as Indexed Alegebras

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Report No. 252
1989

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COMPUTER SCIENCE

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Characterizing Finite Undirected Multigraphs as Indexed Algebras[1]

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Abstract: We study certain basic properties of the set M of finite undirected multigraphs in a category theoretic setting. It turns out to be very natural to consider M as an indexed algebra. The main results of this paper concern representation theorems for indexed algebras. The study is motivated by problems in the structure theory of soliton automata.

1. Introduction

The results reported in this paper originate from a study of structural properties of soliton automata. Soliton automata provide an abstract mathematical model of the switching behaviour of certain complex molecules under the influence of soliton waves [Dv1]. Soliton switching is being discussed as promising component of future computing devices [Ca1].

The states of a soliton automaton are weighted undirected graphs with the same underlying unweighted graph. This underlying graph represents a molecule, and the weight on the edges corresponds to the multiplicity of the respective bonds. A transition is initiated by "inserting" a soliton at some external node and "directing" it to another or even the same external node. Such a transition will cause the weights of the edges to change with each traversal of the soliton. In every graph which occurs as the state of a soliton automaton, the weights of the edges are 1 or 2 only; moreover, for every interior node, the sum of the weights of the edges originating from this node is one more than the number of these edges. Intuitively, a soliton can only move along edges of alternating weights. A soliton path is a path from one exterior node to another or even the same

[1] This research was in part supported by the Natural Science and Engineering Council of Canada, Grant A0243.

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exterior node on which the weights alternate dynamically, that is as the soliton "moves" along the path. For an example see Figure 1. Note that the model of soliton automaton is an abstraction, of course. Hence, the words 'soliton,' 'molecule,' etc. should not be taken in their technical sense without some caution.

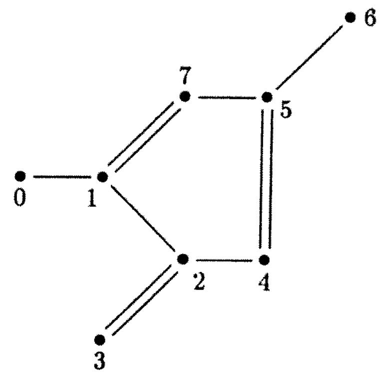

Figure 1. Example of a soliton graph with a cycle. Note that the double lines denote double bonds, not multiple edges.

A soliton automaton is non-deterministic in general. It is deterministic if the state changes are deterministic. Moreover, it is strongly deterministic if in every situtation the soliton has no choice between different paths. Mathematically, strong determinism is the easiest property to deal with. Strongly deterministic soliton automata are characterized in [Da1] as having trees or certain special graphs, called chestnuts, as the connected components of their underlying graphs (assuming the automaton is "reduced" in a sense defined in [Da1]). A chestnut is, essentially, a cycle of even length with all entry points and bifurcations at even distances from each other. The computational power of strongly deterministic soliton automata can, to a certain extent, be described by their transition monoids. These transition monoids turn out to be primitive groups of permutations generated by involutorial elements [Da1]. While it is known that all symmetric groups can be obtained in this way and that, on the other hand, some primitive groups cannot be obtained in this way, the precise characterization of the groups occurring as transition monoids of strongly deterministic soliton automata is still an unsolved problem. Moreover, it is not clear in which way the structure of a soliton graph would be reflected in the structure of the corresponding group of permutations.

For the more general case of deterministic soliton automata, the analysis is even more complicated due to the fact that in the underlying soliton graphs also cycles of odd lengths could exist. So far, only two special situations have been dealt with. First, soliton graphs with a single exterior node—when deterministic—have the cyclic group of order 2 or the trivial group as transition monoid [Da3]. Second, deterministic soliton automata with at most one cycle are studied in [Da2]. Again, some special cases excepted, the transition monoids are primitive groups of permutations generated by involutorial elements. However, the exceptional cases add certain imprimitive groups; the latter can be characterized to some extent as certain groups of monomial matrices over GF(3).

The general analysis of deterministic soliton automata is an open problem and, of course, the questions of above carry over to this more general case.

This was the starting point for the investigation leading to this paper on undirected multigraphs. The theory of multigraphs is to serve as a basis for an algebraic structure theory of soliton automata. This application is presented in [Ba1]. However, beyond this application to soliton automata, the theory of multigraphs is of interest in its own right.

The paper is structured as follows. In Section 2 we provide a very brief review of the notation used. In general, we assume that the reader is familiar with category theory and with the theory of partial algebras (see [ML1], [Bu0], [Re1]). In Section 3, we define the algebra M of (finite) undirected multigraphs. Sections 4–6 build the category theoretic framework in which M can be appropriately described. In the first step, we introduce families of indexed sets as functors from an index category to the category **Set**. Two particular index categories, **U** and **N**, are of special interest with respect to the goal of characterizing M. Section 4 ends with a representation for families of **U**-indexed sets in terms of families of **N**-indexed sets. In Section 5, we introduce indexed algebraic signatures. In particular, we define a **U**-indexed algebraic signature GR and an **N**-indexed algbraic signature Gr and establish the connection between them. Finally, in Section 6 we define indexed algebras, in particular GR-algebras and Gr-algebras. M is obtained as a GR-algebra. The main result of this section is a representation theorem of GR-algebras in terms of Gr-algebras. The paper ends with a few concluding remarks in Section 7.

Considering our original and motivating application, the structure theory of soliton automata, this paper achieves a fundamental step towards a precise treatment of the composition of soliton graphs; no doubt, that quite a few more steps are required. However, we consider this one which clarifies the basic category theoretic issues as quite essential.

2. Review of Notation

The symbols $\mathbb{N}$ and $\mathbb{N}_+$ denote the set of non-negative integers and the set of positive integers, respectively. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$. For a set A, $\mathrm{Fin}(A)$ denotes the set of all finite subsets of A and $\|A\|$ is the cardinality of A. If A and B are sets then $A \ominus B$ is the symmetric difference of A and B, that is, $A \ominus B = (A \setminus B) \cup (B \setminus A)$.

As usual, if $\mathbf{C}$ is a category, $|\mathbf{C}|$ denotes its objects and $\mathbf{C}^{\mathrm{op}}$ is its opposite. We use the symbol $\circ$ to denote composition in diagrammatical order. This includes composition of functors and natural transformations. For a general background in category theory see [ML1], for instance.

3. The Algebra of Undirected Multigraphs

In this section we review some basic notions concerning multigraphs. Moreover, we introduce the algebra of multigraphs and show that it is generated by its constants.

An *undirected multigraph* is a pair $G = (V, E)$ with V the set of *vertices* and E the set of *edges*. An edge $e \in E$ connects two vertices $v_1, v_2 \in V$ without having a direction. The vertices v_1 and v_2 are then said to be the *endpoints* of e. Note that the case of $v_1 = v_2$ is permitted. In this case, e is said to be a *loop at* or *around* v_1. In this paper, we consider finite multigraphs only. Hence, both V and E are assumed to be finite in the sequel.

For a vertex $v \in V$, we define the *degree* $d(v)$ of v to be the number of occurrences of v as an endpoint of an edge in E. Note that, in this definition of the degree of v, every loop contributes 2 occurrences to the count. The vertex v is called *isolated* if $d(v) = 0$, *exterior* if $d(v) = 1$ and *interior* otherwise. Let $\text{ext}(G)$ be the set of exterior vertices of G. An edge e is said to be an *exterior edge* if one of its endpoints is exterior.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, , E_2)$ be two undirected multigraphs. They are said to be *isomorphic* if $\text{ext}(G_1) = \text{ext}(G_2)$ and there are bijections φ of V_1 onto V_2 and η of E_1 onto E_2 such that the restriction of φ to $\text{ext}(G_1)$ is the identity mapping and φ and η satisfy the usual conditions for graph isomorphisms. Similarly, G_1 and G_2 are considered *disjoint* if $V_1 \cap V_2 \subseteq \text{ext}(G_1) \cap \text{ext}(G_2)$ and $E_1 \cap E_2 = \emptyset$.

Thus, isomorphic undirected multigraphs always have the same set of exterior vertices. We can also think of the exterior vertices of an undirected multigraph G as being labelled by distinct elements of a finite set A; in this case we would say that G *is of sort* A. Then an isomorphism between two undirected multigraphs of the same sort would have to preserve the labelling of the exterior vertices. In the sequel, when we say that G is of sort A or labelled by A we assume without special mention that the labelling induces a bijection of A onto $\text{ext}(G)$.

Consider a set U of finite sets which is closed under the operation of symmetric difference and contains the set $\{[n] \mid n \in \mathbb{N}\}$. It follows that the set $\text{Fin}(\mathbb{N}_+)$ is the smallest possible choice for U. We now define a U-sorted algebra M on undirected multigraphs as follows.

(i) For the sort $A \in U$, the underlying set M_A of M is the set of all isomorphism classes of undirected multigraphs of sort A, that is, the set of all isomorphism classes of those undirected multigraphs whose exterior vertices belong to A or, equivalently, are labelled by the elements of A.

In the sequel, we simplify the terminology by identifying isomorphism classes of undirected multigraphs with appropriately chosen representatives. Note that, by this convention, any finite number of undirected multigraphs can be assumed to be mutually disjoint.

(ii) M is equipped with a binary operation "$\cdot$" (or rather, a collection of binary operations) called *composition* such that

$$\cdot : M_{A_1} \times M_{A_2} \to M_{A_1 \ominus A_2}$$

for all $A_1, A_2 \in U$. Intuitively, the composition is that of "pasting" two undirected multigraphs together at the exterior vertices which they have in common. Formally, let $G_1 = (V_1, E_1) \in M_{A_1}$ and $G_2 = (V_2, E_2) \in M_{A_2}$ be two undirected multigraphs. Without loss of generality, we may assume that G_1 and G_2 are disjoint. The composite $G_1 \cdot G_2$ is obtained using the following steps:

(1) Let $L = (V_1 \cup V_2, E_1 \cup E_2)$.

(2) For any two vertices $v_1, v_2 \in (V_1 \cup V_2) \setminus (A_1 \cap A_2)$ which are connected in L by a path consisting only of exterior edges with endpoints in $A_1 \cap A_2$, add a new edge connecting v_1 and v_2. Let L' be the resulting undirected multigraph.

(3) For each cycle in L' containing only vertices in $A_1 \cap A_2$, add a new isolated vertex. Let L'' be the resulting undirected multigraph.

Observe that a cycle considered in step (3) has no multiple edges and its length is always even.

(4) In L'', delete the vertices in $A_1 \cap A_2$ and all edges with endpoints in this set. The resulting undirected multigraph is defined to be $G_1 \cdot G_2$.

For given sorts A_1, A_2, the restriction of the composition to $M_{A_1} \times M_{A_2}$ is denoted by $[A_1, A_2]$. An example of the composition is provided in Figure 2.

(iii) M has the following infinite set of constants: $0 \in M_\emptyset$ denoting the empty graph; $1_A \in M_A$ for each two-element set A; $n_A \in M_A$ for each $n \in N$ and $A \in U$ with $\|A\| = n$. The interpretation of the constants is shown in Figure 3.

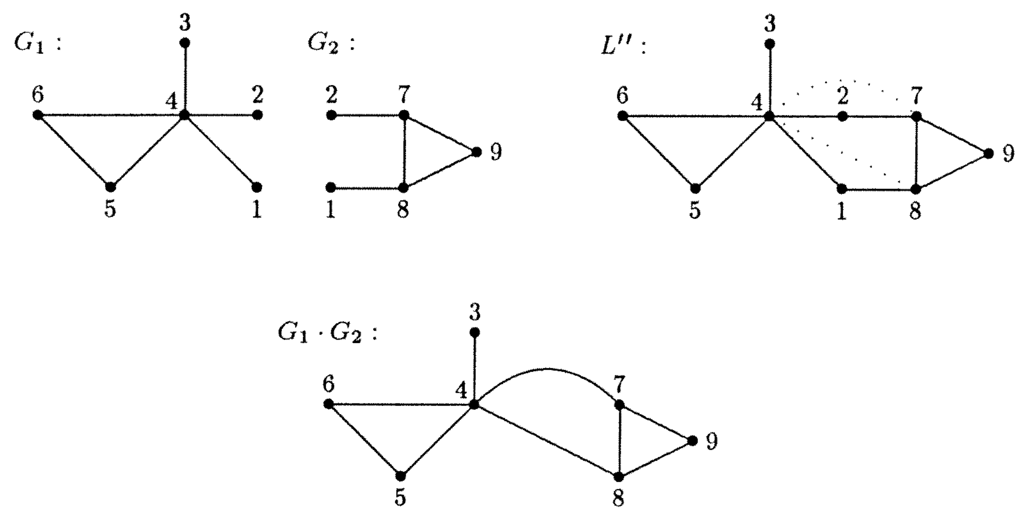

Figure 2. Example showing the 4 steps of the composition in M. The edges inserted in step (2) are indicated as dotted lines. In this example, step (3) does not introduce any new vertices.

Note that the philosophy of defining the composite of two undirected multigraphs is exactly the same as that followed in the definition of the usual flowchart scheme operations of composition, sum (tupling), and feedback (iteration) (for example, see [Bl1], [Ba2]). In fact, our aim was to define the algebra M (or rather its contraction as defined below) to be the undirected and symmetric counterpart of the algebra of flowchart schemes.

Proposition 3.1 *The algebra M is generated by its constants.*

Proof: Indeed, every multigraph G can be assembled from its interior vertices of degree $n \geq 2$, each considered as an instance of the constant n_A for a suitable $A \in U$, and some instances of the constants 1_A using only composition, which in this case can be either "parallel" or "sequential." $\square$

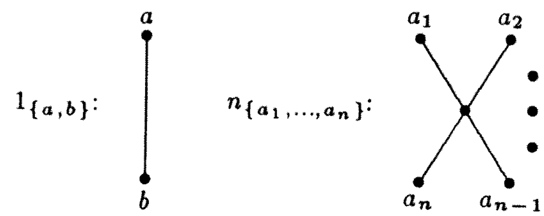

Figure 3. Interpretation of the constants of M.

To illustrate the proof of Proposition 3.1 we indicate how two special undirected multigraphs can be obtained, the undirected multigraph consisting of a single isolated vertex and the one consisting of a single vertex with a loop around it. The former is obtained as $1_{\{a,b\}} \cdot 1_{\{a,b\}}$; in this case, step 3 of the construction introduces the isolated vertex while step 4 removes all other vertices. The latter is given by $2_{\{a,b\}} \cdot 1_{\{a,b\}}$; here step 2 introduces the loop around the interior vertex of $2_{\{a,b\}}$ while step 4 removes all other vertices and edges.

4. Indexed Families of Sets and M

In the remainder of this paper we investigate the basic properties of the algebra M of undirected multigraphs in terms of category theory.

The main results of this section concern a rigorous treatment of the relabelling of exterior vertices required to construct the composition of undirected multigraphs in M. In particular, we give a back-and-forth translation of labels between U and the set $\{[n] \mid n \in \mathbb{N}\}$. This issue is important, as depending on the situation the preferred label set will change. A representation theorem is proved which affords the connection.

Recall that an *indexed category* $\mathbf{C}$ over an *index category* $\mathbf{Ind}$ is a functor $\mathbf{Ind}^{\mathrm{op}} \to \mathbf{Cat}$ (see [Bul]). Given an index $i \in |\mathbf{Ind}|$, we write $\mathbf{C}_i$ for the category $\mathbf{C}i$, and given an index morphism $\sigma : i \to j$, we write $\mathbf{C}_\sigma$ for the functor $\mathbf{C}\sigma : \mathbf{C}i \to \mathbf{C}j$. We also adopt from [Bul] the construction of flattening an indexed category by means of the Grothendieck fibration [Grl]. For an indexed category $\mathbf{C} : \mathbf{Ind}^{\mathrm{op}} \to \mathbf{Cat}$, its flattening $\mathbf{Flat}(\mathbf{C})$ is defined as follows.

- *Objects:* pairs $\langle i, a \rangle \in |\mathbf{Ind}| \times |\mathbf{C}_i|$.

- *Morphisms:* pairs $\langle\sigma, f\rangle : \langle i, a\rangle \to \langle j, b\rangle$ where $\sigma : i \to j$ is a morphism in **Ind** and $f : a \to \mathbf{C}_\sigma b$ is a morphism in $\mathbf{C}_i$.
- *Composition of morphisms:* for morphisms $\langle\sigma, f\rangle : \langle i, a\rangle \to \langle j, b\rangle$ and $\langle\varrho, g\rangle : \langle j, b\rangle \to \langle k, c\rangle$ in $\mathbf{Flat}(\mathbf{C})$, let

$$\langle\sigma, f\rangle \circ \langle\varrho, g\rangle = \langle\sigma \circ \varrho, f \circ \mathbf{C}_\sigma g\rangle : \langle i, a\rangle \to \langle k, c\rangle.$$

In particular, note that, given any index category **Ind**, every functor $\mathbf{C} : \mathbf{Ind}^{\mathrm{op}} \to \mathbf{Set}$ defines an indexed category—also denoted by **C**—in which $\mathbf{C}_i$ is the set $\mathbf{C}i$ considered as a discrete category and, for an index morphism $\sigma : i \to j$, $\mathbf{C}_\sigma$ is $\mathbf{C}\sigma$ viewed as a functor between the discrete categories $\mathbf{C}_i$ and $\mathbf{C}_j$. Although it would seem natural, we do not refer to such a functor as a "family of indexed sets over **Ind**." Instead, we reserve this term for covariant functors $\mathbf{Ind} \to \mathbf{Set}$.

Definition 4.1 Let **Ind** be a category. An **Ind**-*indexed family of sets* is a functor $\mathbf{C} : \mathbf{Ind} \to \mathbf{Set}$.

Thus, an **Ind**-indexed family of sets is an $|\mathbf{Ind}|$-sorted family of sets in the usual sense, equipped with some mappings between its members corresponding to the morphisms in **Ind**. Although the indexed family of sets **C** is not an indexed category we nevertheless use the notation $\mathbf{C}_i$ and $\mathbf{C}_\sigma$ for $\mathbf{C}i$ and $\mathbf{C}\sigma$ as long as there is no risk of confusion.

Following the pattern of the first three examples in [Bul], we define the indexed category of indexed algebras, and characterize the graph algebra M as a special object in this category. We first introduce the indexed category

$$\mathbf{ISET} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{Cat}$$

of indexed sets. The construction is analogous to that of the indexed category **SSET** in [Bul], Example 1.

For a category **S**, define $\mathbf{ISET_S}$ as the category $[\mathbf{S} \to \mathbf{Set}]$ of functors of **S** into **Set**. Thus, $\mathbf{ISET_S}$ can be considered as the category of all **S**-indexed family of sets. For a functor $\mathbf{\Lambda} : \mathbf{S} \to \mathbf{S}'$, the functor $\mathbf{ISET_\Lambda} : \mathbf{ISET_{S'}} \to \mathbf{ISET_S}$ is defined as follows.

- *On objects:* For $X \in |\mathbf{ISET_{S'}}|$, that is, for a functor $X : \mathbf{S}' \to \mathbf{Set}$, let

$$\mathbf{ISET_\Lambda} X = \mathbf{\Lambda} \circ X : \mathbf{S} \to \mathbf{Set}.$$

- *On morphisms:* For a natural transformation $\mu : X \to Y$ between functors $X, Y \in |\mathbf{ISET_{S'}}|$, let

$$\mathbf{ISET_\Lambda} \mu = \mathbf{\Lambda} \circ \mu : \mathbf{\Lambda} \circ X \to \mathbf{\Lambda} \circ Y.$$

The construction of $\mathbf{Flat}(\mathbf{ISET})$ is left to the reader.

In a similar fashion, one defines the indexed category $\mathbf{ISET}^*$ of indexed pointed sets. In the construction, one replaces the category **Set** by the category $\mathbf{Set}^*$ of pointed sets with point preserving mappings as morphisms (see [ML1]).

In this paper, the follwing two index categories **U** and **N** are of particular interest.

- Given a set U as in section **3**, the category **U** has U as its set of objects. The morphisms are the bijections between sets in U with the identities and the composition defined as usual.

- The category $\mathbf{N}$ has $\mathbb{N}$ as its set of objects[4]. The morphisms are the n-ary permutations for each $n \in \mathbb{N}$. Again the identities and the composition are defined as usual.

Given a family

$$\Psi = \{\psi_A : A \to [\|A\|] \mid A \in U\}$$

of bijections, we can define a functor $\mathbf{\Psi} : \mathbf{U} \to \mathbf{N}$ in such a way that

$$\mathbf{\Psi}A = \|A\| \quad \text{and} \quad \mathbf{\Psi}\alpha = \psi_A^{-1} \circ \alpha \circ \psi_B$$

holds for every $A, B \in U$ and every $\alpha : A \to B$ in $\mathbf{U}$. If $\alpha : A \to B$ and $\beta : B \to C$, then

$$\begin{aligned}\mathbf{\Psi}\alpha \circ \mathbf{\Psi}\beta &= \psi_A^{-1} \circ \alpha \circ \psi_B \circ \psi_B^{-1} \circ \beta \circ \psi_C \\ &= \psi_A^{-1} \circ (\alpha \circ \beta) \circ \psi_C = \mathbf{\Psi}(\alpha \circ \beta).\end{aligned}$$

This shows that $\mathbf{\Psi}$ is indeed a functor.

The carrier of the algebra M of undirected multigraphs is a family of sets indexed by $\mathbf{U}$. Hence, M can be considered as a functor $M : \mathbf{U} \to \mathbf{Set}$ as follows: With each $A \in U = |\mathbf{U}|$, M associates the set M_A of undirected multigraphs whose exterior vertices are labelled by A. With every morphism $\alpha : A \to B$ of $\mathbf{U}$, M associates the bijection $M_\alpha : M_A \to M_B$ defined as the relabelling of the exterior vertices of undirected multigraphs according to α.

The restriction of $\mathbf{U}$ to the objects $\{[n] \mid n \in \mathbb{N}\}$ is isomorphic with $\mathbf{N}$. Therefore, one can define the family of $\mathbf{N}$-indexed sets $\{M_{[n]} \mid n \in \mathbb{N}\}$, that is, the functor $\mathbf{Con}M : \mathbf{N} \to \mathbf{Set}$. This functor is referred to as the *contraction* of M. We generalize this idea to obtain a *contracting functor* $\mathbf{Con} : \mathbf{ISET_U} \to \mathbf{ISET_N}$, which maps each object (functor) and each morphism (natural transformation) to its restriction to $\mathbf{N}$, that is, to the set $\{[n] \mid n \in \mathbb{N}\}$). On the other hand, for each functor $\mathbf{\Psi}$ as defined above, we have the functor $\mathbf{ISET_\Psi} : \mathbf{ISET_N} \to \mathbf{ISET_U}$ in the opposite direction.

Theorem 4.2 *For every family $\Psi = \{\psi_A : A \to [\|A\|] \mid A \in U\}$ of bijections one has*

$$id_{[\mathbf{U} \to \mathbf{Set}]} \cong \mathbf{Con} \circ \mathbf{ISET_\Psi}$$

and

$$id_{[\mathbf{N} \to \mathbf{Set}]} \cong \mathbf{ISET_\Psi} \circ \mathbf{Con}$$

using appropriate natural isomorphisms.

[4] As a matter of notational convenience, we frequently identify $n \in |\mathbf{N}| = \mathbb{N}$ with $[n] \in |\mathbf{U}|$. Thus $\mathbf{N}$ can be considered as a subcategory of $\mathbf{U}$.

$$\begin{tikzcd}
FA \arrow[rrr, "(\mu F)A = F\psi_A"] \arrow[d, "F\alpha"'] & & & (\mathbf{ISET}_{\Psi}(\mathbf{Con}F))A = F[||A||] \arrow[d, "(\mathbf{ISET}_{\Psi}(\mathbf{Con}F))\alpha"] \\
FB \arrow[rrr, "(\mu F)B = F\psi_B"'] & & & (\mathbf{ISET}_{\Psi}(\mathbf{Con}F))B = F[||B||]
\end{tikzcd}$$

Figure 4. Diagram for the firsts part of the proof of Theorem 4.2.

Proof: For the first statement, we need to find a natural isomorphism $\mu : id_{[\mathbf{U}\to\mathbf{Set}]} \to \mathbf{Con} \circ \mathbf{ISET}_{\Psi}$. For this purpose, we define $\mu F : F \to \mathbf{ISET}_{\Psi}(\mathbf{Con}F)$ for each functor $F : \mathbf{U} \to \mathbf{Set}$ as the second component of a suitable morphism

$$\langle \Psi, \mu F\rangle : \langle \mathbf{U}, F\rangle \to \langle \mathbf{N}, \mathbf{Con}F\rangle$$

in the flattened category $\mathbf{Flat}(\mathbf{ISET})$. By definition,

$$\mathbf{ISET}_{\Psi}(\mathbf{Con}F) = \Psi \circ \mathbf{Con}F.$$

Hence, for every $A \in U$,

$$(\mathbf{ISET}_{\Psi}(\mathbf{Con}F))A = (\mathbf{Con}F)\Psi A = (\mathbf{Con}F)||A|| = F[||A||].$$

Therefore, we can define $(\mu F)A$ to be $F\psi_A$. Now consider a morphism $\alpha : A \to B$ in $\mathbf{U}$. Then

$$\begin{aligned}(\mathbf{ISET}_{\Psi}(\mathbf{Con}F))\alpha &= (\mathbf{Con}F)\Psi\alpha \\ &= (\mathbf{Con}F)(\psi_A^{-1} \circ \alpha \circ \psi_B) = F(\psi_A^{-1} \circ \alpha \circ \psi_B) \\ &= F\psi_A^{-1} \circ F\alpha \circ F\psi_B.\end{aligned}$$

The diagram corresponding to these equations is given in Figure 4. This proves that μ is a natural isomorphism.

For the second part, let $\Psi_0 = \{\psi_A^0 : A \to [||A||] \mid A \in U\}$ be a family of bijections subject to the condition that ψ_A^0 is the identical n-ary permutation whenever $A = [n]$ for some $n \in \mathbb{N}$. Essentially, $\mathbf{ISET}_{\Psi_0}$ affords an embedding of $\mathbf{ISET}_{\mathbf{N}}$ in $\mathbf{ISET}_{\mathbf{U}}$. As an immediate consequence of the definitions one has

$$id_{[\mathbf{N}\to\mathbf{Set}]} = \mathbf{ISET}_{\Psi_0} \circ \mathbf{Con}.$$

Now let $\Psi = \{\psi_A : A \to [||A||] \mid A \in U\}$ be an arbitrary family of bijections. One verifies that

$$\begin{aligned}id_{[\mathbf{N}\to\mathbf{Set}]} &= \mathbf{ISET}_{\Psi_0} \circ \mathbf{Con} \\ &\cong \mathbf{ISET}_{\Psi_0} \circ \mathbf{Con} \circ \mathbf{ISET}_{\Psi} \circ \mathbf{Con} = \mathbf{ISET}_{\Psi} \circ \mathbf{Con}\end{aligned}$$

using the natural isomorphism $\nu = \mathbf{ISET}_{\Psi_0} \circ \mu \circ \mathbf{Con}$. $\square$

By Theorem 4.2, the functor $\mathbf{ISET}_{\Psi}$ is a kind of inverse of the contraction functor **Con**. From this point of view it is a bit inconvenient that the "inverse" would vary with the choice of the functor Ψ. Indeed, occasionally we would prefer to have a "canonical inverse" $\mathbf{Con}^{-1}$. Such a functor $\mathbf{Con}^{-1}$ is constructed next. As a preparation, we define a the family $Ord : \mathbf{U} \to \mathbf{Set}$ of $\mathbf{U}$-indexed sets as follows.

- *On objects:* For $A \in \mathbf{U}$, $Ord_A = OrdA$ is the set of all bijections of $[\|A\|]$ onto A.
- *On morphisms:* For a morphism $\alpha : A \to B$ in $\mathbf{U}$, $Ord_\alpha : Ord_A \to Ord_B$ is the bijection which maps $\varphi \in Ord_A$ to $\varphi \circ \alpha$.

Obviously, Ord is a functor. It can be equipped with the multigraph operations $[A, B]$ as follows. Consider $A, B \in \mathbf{U}$ with $\|A\| = n$, $\|B\| = m$, and $\|A \cap B\| = q$, and let $\varphi \in Ord_A$ and $\chi \in Ord_B$. Enumerate the elements of $\varphi^{-1}(A \setminus B)$ and $\chi^{-1}(B \setminus A)$ as $l_1 < l_2 < \ldots < l_{n-q}$ and $k_1 < k_2 < \ldots < k_{m-q}$, respectively. Now define the permutation $(\varphi[A, B]\chi)$ of $[n + m - 2q]$ by

$$(\varphi[A, B]\chi)(i) = \begin{cases} \varphi(l_i), & \text{if } i \in [n-q], \\ \chi(k_j), & \text{if } i = n - q + j \text{ for some } j \in [m-q]. \end{cases}$$

Consider two morphisms $\alpha : A \to A'$ and $\beta : B \to B'$ in $\mathbf{U}$. The morphisms α and β are said to be *compatible* if, for all $a \in A$ and $b \in B$,

$$\alpha(a) = \beta(b) \iff a = b \in A \cap B.$$

For compatible morphisms α and β, we define $\alpha \ominus \beta : A \ominus B \to A' \ominus B'$ by

$$(\alpha \ominus \beta)(x) = \begin{cases} \alpha(x), & \text{if } x \in A \\ \beta(x), & \text{if } x \in B. \end{cases}$$

Clearly, $\alpha \ominus \beta$ is a morphism in $\mathbf{U}$.

One verifies by some computation that this composition obeys certain natural "relabelling rules." It is shown later, that these rules also obtain in M.

Proposition 4.3 *Let $\alpha : A \to A'$ and $\beta : B \to B'$ be compatible morphisms in $\mathbf{U}$. The diagram*

$$\begin{tikzcd} Ord_A \times Ord_B \arrow[r, "{[A,B]}"] \arrow[d, "Ord_\alpha \times Ord_\beta"'] & Ord_{A \ominus B} \arrow[d, "Ord_{\alpha \ominus \beta}"] \\ Ord_{A'} \times Ord_{B'} \arrow[r, "{[A',B']}"'] & Ord_{A' \ominus B'} \end{tikzcd}$$

commutes.

We now turn to defining $\mathbf{Con}^{-1}$. For any $\mathbf{N}$-indexed family of sets $G : \mathbf{N} \to \mathbf{Set}$, we consider the functor $\mathbf{Con}^{-1}G : \mathbf{U} \to \mathbf{Set}$ given as follows.

- *On objects:* For $A \in U$ let $(\mathbf{Con}^{-1}G)A$ be the quotient of $G[\|A\|] \times Ord_A$ with respect to the equivalence relation $\sim$ given by

$$(a, \varphi) \sim (b, \chi) \iff b = G(\varphi \circ \chi^{-1})(a).$$

  For an interpretation of this equivalence relation, an undirected multigraph of sort A can be regarded as being of sort $[\|A\|]$ together with an ordering of its exterior vertices. In this sense, two multigraphs are equivalent if and only if they differ only in the ordering of their exterior vertices.
- *On morphisms:* For a morphism $\alpha : A \to B$ in $\mathbf{U}$, let $(\mathbf{Con}^{-1}G)\alpha$ be the bijection of $(\mathbf{Con}^{-1}G)A$ onto $(\mathbf{Con}^{-1}G)B$ which maps the equivalence class of (a, φ) onto the equivalence class of $(a, \varphi \circ \alpha)$.

The correctness of this definition is easily verified. Moreover, one extends $\mathbf{Con}^{-1}$ to a functor in the obvious fashion.

We now establish the connection to our earlier results. Given a family of bijections $\Psi = \{\psi_A : A \to [\|A\|] \mid A \in U\}$, one obtains a natural isomorphism between $\mathbf{ISET}_\Psi G$ and $\mathbf{Con}^{-1}G$ as follows.

- For $A \in U$, represent the equivalence classes in $(\mathbf{Con}^{-1}G)A$ by pairs of the form (a, ψ_A^{-1}). Note that every equivalence class contains a pair of this form; moreover, this pair is actually unique as $(a, \psi_A^{-1}) \sim (b, \psi_A^{-1})$ implies $b = G(\psi_A^{-1} \circ \psi_A)(a) = a$. In this way,

$$(\mathbf{ISET}_\Psi G)A = G[\|A\|] \cong (\mathbf{Con}^{-1}G)A.$$

To prove that $\cong$ is a natural transformation, let $\alpha : A \to B$ be a morphism in $\mathbf{U}$. Then, for all $a \in G[\|A\|]$,

$$((\mathbf{ISET}_\Psi G)\alpha)(a) = G(\psi_A^{-1} \circ \alpha \circ \psi_B)(a) \in G[\|B\|]$$

and

$$((\mathbf{Con}^{-1}G)\alpha)(a, \psi_A^{-1}) \sim (a, \psi_A^{-1} \circ \alpha) \sim (G(\psi_A^{-1} \circ \alpha \circ \psi_B)(a), \psi_B^{-1}).$$

This construction is natural in G. Hence, it yields a natural isomorphism between $\mathbf{ISET}_\Psi$ and $\mathbf{Con}^{-1}$. Therefore, in Theorem 4.2 we can replace $\mathbf{ISET}_\Psi$ by $\mathbf{Con}^{-1}$. Denoting $\mathbf{Con}^{-1}G$ by $G[Ord]$, we have the following representation theorem for families of $\mathbf{U}$-indexed sets.

Theorem 4.4 *Every family F of $\mathbf{U}$-indexed sets is naturally isomorphic to $G[Ord]$ for a suitable family G of $\mathbf{N}$-indexed sets.*

Proof: Put $G = \mathbf{Con}F$. $\square$

5. Indexed Algebraic Signatures and M

In this section we introduce the indexed category of indexed algebraic signatures **ISIG**. In order to deal with the algebra M of undirected multigraphs in this setting, we then define the **U**-indexed algebraic signature GR and the **N**-indexed algebraic signature Gr. The latter is obtained as a kind of contraction of the former. The main result of this section describes the connection between these two functors.

Definition 5.1 Let $\mathbf{S}$ be a category. Its *rank category* $\mathbf{S}^+$ is the coproduct of the categories $\mathbf{S}, \mathbf{S}^2, \ldots$ in **Cat**. The objects of $\mathbf{S}^+$ are tuples of the form

$$\langle s_1, \ldots, s_n; s_{n+1} \rangle$$

with $s_i \in |\mathbf{S}|$ for $i = 1, \ldots, n+1$. The morphisms

$$\langle s_1, \ldots, s_n; s_{n+1} \rangle \to \langle s'_1, \ldots, s'_n; s'_{n+1} \rangle$$

are tuples

$$\langle \alpha_1, \ldots, \alpha_n; \alpha_{n+1} \rangle$$

of morphisms $\alpha_i : s_i \to s'_i$ in $\mathbf{S}$ for $i \in [n+1]$. The objects of $\mathbf{S}^+$ are called *ranks*.[5]

Definition 5.2 Let $\mathbf{S}$ be a category. An $\mathbf{S}$*-indexed algebraic signature* is a functor $\Sigma : \mathbf{S}^+ \to \mathbf{Set}^*$.

Let Σ be an $\mathbf{S}$-indexed algebraic signature. Intuitively, with each rank

$$\langle s_1, \ldots, s_n; s_{n+1} \rangle$$

the functor Σ associates the set of n-ary operation symbols of this rank, that is, the elements of $\Sigma\langle s_1, \ldots, s_n; s_{n+1} \rangle$ which are different from the point $*$. We interpret the morphisms of $\mathbf{S}^+$ as rank relabellings. Suppose that $\Diamond$ is an n-ary operation symbol of rank $r = \langle s_1, \ldots, s_n; s_{n+1} \rangle$, that is, $\Diamond \in \Sigma r \setminus \{*\}$ and that $\varrho : r \to r'$ is a morphism in $\mathbf{S}^+$. Then the functor Σ specifies whether the relabelling according to ϱ is permitted or not: ϱ is said to be *allowable* or *permitted* for $\Diamond$ in Σ if and only if $(\Sigma\varrho)\Diamond \neq *$. We use the notation $\varrho : \Diamond \mapsto \Diamond'$ to mean that ϱ is allowable for $\Diamond$ and that $(\Sigma\varrho)\Diamond = \Diamond'$. In this case we would also say that the *replacement* $\Diamond \mapsto \Diamond'$ *is permitted by* ϱ *in* Σ.

In a manner simmilar to [Bul], we use the extension of the map $\mathbf{S} \to \mathbf{S}^+$ to a functor $(\_)^+ : \mathbf{Cat} \to \mathbf{Cat}$, to define the *indexed category of indexed algebraic signatures* as

$$\mathbf{ISIG} = ((\_)^+)^{\mathrm{op}} \circ \mathbf{ISET}^* : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{Cat}.$$

Flattening **ISIG** results the category **Flat(ISIG)**, whose objects are pairs $\langle \mathbf{S}, \Sigma \rangle$, where $\mathbf{S}$ is a category and Σ is a functor $\mathbf{S}^+ \to \mathbf{Set}^*$. A morphism from $\langle \mathbf{S}, \Sigma \rangle$ to $\langle \mathbf{S}', \Sigma' \rangle$ is a pair $\langle \mathbf{\Lambda}, \omega \rangle$, where $\mathbf{\Lambda} : \mathbf{S} \to \mathbf{S}'$ is an index renaming functor and $\omega : \Sigma \to \mathbf{ISIG}_{\mathbf{\Lambda}}\Sigma'$

[5] To keep the notation simple in the case of ranks of the form $\langle ; s \rangle$ for $s \in S$, we often write s instead.

is a natural transformation. Note that the operation symbols of rank $\langle s_1, \ldots, s_n; s_{n+1}\rangle$ of $\mathbf{ISIG}_\Lambda \Sigma'$ are precisely the operation symbols of Σ' whose rank is $\langle \Lambda s_1, \ldots, \Lambda s_n; \Lambda s_{n+1}\rangle$. A relabelling ϱ is permitted in $\mathbf{ISIG}_\Lambda \Sigma'$ for such an operation symbol if and only if $\Lambda\varrho$ is permitted for it in Σ'.

The natural transformation $\omega : \Sigma \rightarrow \mathbf{ISIG}_\Lambda \Sigma'$ is said to be *non-degenerate* if for no $r \in |\mathbf{S}^+|$ the arrow ωr sends any operation symbol to $*$. Let **ISig** denote the subcategory of **Flat(ISIG)** in which the second components of the morphisms are non-degenerate. Clearly, the composition of such morphisms is again of this kind which proves that **ISig** is indeed a subcategory.

We now consider this construction for our special index categories **U** and **N**. First we define a **U**-indexed algebraic signature GR as follows.

- *Operation symbols:* For each triple $r = \langle A, B; C\rangle \in |\mathbf{U}^3| = U^3$, $GR(r)$ contains an operation symbol $[A, B]$ of rank r if and only if $C = A \ominus B$.
- *Relabelling rules:* A rank relabelling $\langle \alpha, \beta; \gamma\rangle : \langle A, B; A \ominus B\rangle \rightarrow \langle A', B'; A' \ominus B'\rangle$is permitted for $[A, B]$ and yields $[A', B']$ if and only if α and β are compatible and $\gamma = \alpha \ominus \beta$.
- *Constants:* There is a unique constant 0 of rank $\emptyset$, that is, of rank $\langle ; \emptyset\rangle$. The relabelling $id_\emptyset$ is permitted for 0 by definition.

  For each two-element set $A \in U$, there are two constants 1_A and 2_A of rank A. Every relabelling $\alpha : A \rightarrow A'$ is permitted resulting in $1_{A'}$ and $2_{A'}$, respectively.

  For each n-element set $A \in U$, $n \geq 3$, there is unique constant n_A of rank A. Every relabelling $\alpha : A \rightarrow A'$ is permitted resulting in $n_{A'}$.

Note that GR is the signature of our graph algebra M, enriched with the natural relabelling rules. It is easy to see that these rules make GR a functor $\mathbf{U}^+ \rightarrow \mathbf{Set}^*$.

Next we define an **N**-indexed algebraic signature Gr by "contracting" the signature GR.

- *Operation symbols:* Let $r = \langle n, m; n + m - 2q\rangle \in \mathbf{N}^3$ with $q \leq \min(n, m)$. Then $Gr(r)$ consists of the operation symbols $[\pi, \xi]$ with π and ξ satisfying the following conditions:
  - π is a bijection between sets, $\pi : x \rightarrow y$, where x and y are subsets of $[n]$ and $[m]$, respectively, with $\|x\| = \|y\| = q$.
  - ξ is a permutation of $[n + m - 2q]$.

Before defining the relabelling rules and the constants, we explain the intended interpretation of the operation symbols.

Consider $[\pi, \xi] \in Gr(r)$ where $r = \langle n, m; n + m - 2q\rangle$ with $q \leq \min(n, m)$ and $\pi : x \rightarrow y$. Let $\varphi_x : x \rightarrow [q]$ and $\varphi_y : y \rightarrow [q]$ be bijections such that φ_x is monotonic and $\pi = \varphi_x \circ \varphi_y^{-1}$. Let $\varphi'_x : [n] \setminus x \rightarrow [n - q]$ and $\varphi'_y : [m] \setminus y \rightarrow [m - q]$ be the unique monotonic bijections[6]. Let $A = [q] + [n - q]$ and $B = [q] + [m - q]$ be "new" sets, that is, sets such that neither they themselves nor any of their non-empty subsets is in U. Let $\varphi_A = \varphi_x + \varphi'_x : [n] \rightarrow A$, and $\varphi_B = \varphi_y + \varphi'_y : [m] \rightarrow B$. The situation is illustrated in Figure 3. We extend **U** by adding A, B, and $A \ominus B$ as new objects and by adding all necessary morphisms. Let $\mathbf{U}'$ be this extension of **U**. GR has a unique extension to $\mathbf{U}'$.

[6] Strictly speaking, one would have to use an isomorphic copy of $[m - q]$ in order to ensure that $A \cap B = [q]$ for the sets constructed in the sequel.

Finally, let $\varphi_{A \ominus B} : [n+m-2q] \rightarrow A \ominus B$ be the inverse of the bijection[7] $(id_{[n-q]} + id_{[m-q]}) \circ \xi$. Note that this construction provides a one-to-one correspondence between the operation symbols $[\pi, \xi]$ and the triples $(\varphi_A, \varphi_B, \varphi_{A \ominus B})$. Now, the operation symbol $[\pi, \xi]$ can be interpreted as an operation on undirected multigraphs $G \in M_{[n]}$ and $H \in M_{[m]}$ as follows. First, relabel G and H according to φ_A and φ_B to obtain graphs $G' \in M_A$ and $H' \in M_B$. Next, compute $G'[A, B]H'$ in M as if A and B were in U. Finally, relabel the result according to $\varphi_{A \ominus B}^{-1}$. The correctness of the resulting labelling is stated in the following lemma.

Lemma 5.3 $\xi = (\varphi_A[A, B]\varphi_B) \circ \varphi_{A \ominus B}^{-1}$.

Proof: By the definition of $[A, B]$ in *Ord* one has

$$(\varphi_A[A, B]\varphi_B) \circ (id_{[n-q]} + id_{[m-q]}) = id_{[m+n-2q]}.$$

Given the definition of $\varphi_{A \ominus B}$, this implies the statement of the lemma. □

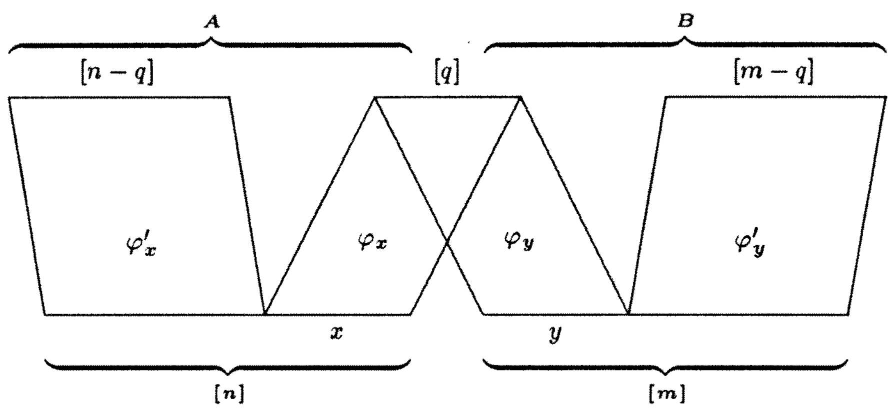

Figure 5. Illustration of φ_x, φ_y, φ'_x, and φ'_y for the interpretation of the operation symbol $[\pi, \xi]$.

We now continue the definition of the **N**-indexed algebraic signature *Gr*.

- *Relabelling rules:* Let

$$\langle \alpha, \beta; \gamma \rangle : \langle n, m; n+m-2q \rangle \rightarrow \langle n, m; n+m-2q \rangle$$

be a morphism in $\mathbf{N}^3$, that is, a rank relabelling. Let $[\pi, \xi]$ and $[\pi', \xi']$ be operation symbols of rank $\langle n, m; n+m-2q \rangle$. In order to determine whether the replacement

[7] To be exact, $id_{[n-q]} + id_{[m-q]} : [n-q] + [m-q] \rightarrow [n+m-2q]$ here, that is, the object $[n-q] + [m-q]$ is identified with $[n+m-2q]$.

$[\pi, \xi] \mapsto [\pi', \xi']$ is permitted by $\langle \alpha, \beta; \gamma \rangle$ in Gr we again look "back" into GR. Again consider the new objects A, B, and $A \ominus B$ introduced above for $[\pi, \xi]$. In the same manner, let A', B', and $A' \ominus B'$ be new objects introduced for $[\pi', \xi']$. Let $(\varphi_A, \varphi_B, \varphi_{A \ominus B})$ and $(\varphi_{A'}, \varphi_{B'}, \varphi_{A' \ominus B'})$ be the triples corresponding to $[\pi, \xi]$ and $[\pi', \xi']$, respectively, as described above. The permutations α, β and γ determine the bijections $\alpha^* : A \to A'$, $\beta^* : B \to B'$, and $\gamma^* : A \ominus B \to A' \ominus B'$ for which

$$\varphi_A \circ \alpha^* = \alpha \circ \varphi_{A'}, \quad \varphi_B \circ \beta^* = \beta \circ \varphi_{B'},$$

and

$$\varphi_{A \ominus B} \circ \gamma^* = \gamma \circ \varphi_{A' \ominus B'}$$

holds. The replacement $[\pi, \xi] \mapsto [\pi', \xi']$ is permitted by $\langle \alpha, \beta; \gamma \rangle$ in Gr if and only if $\langle \alpha^*, \beta^*; \gamma^* \rangle$ is allowable for $[A, B]$ in GR, that is, if and only if α^* and β^* are compatible and $\gamma^* = \alpha^* \ominus \beta^*$. It is easy to see that, when the replacement is permitted, $\langle \alpha, \beta; \gamma \rangle$ and $[\pi, \xi]$ together determine $[\pi', \xi']$ uniquely; in this case, one defines $(Gr\langle \alpha, \beta; \gamma \rangle)[\pi, \xi] = [\pi', \xi']$. Otherwise, $(Gr\langle \alpha, \beta; \gamma \rangle)[\pi, \xi] = *$. Clearly, this defines $Gr\langle \alpha, \beta; \gamma \rangle$ as a morphism in $\mathbf{Set}^*$.

- *Constants:* There is a unique constant 0 of rank 0. The relabelling id_0 is the only one permitted for 0 yielding 0.

  There are two constants 1 and 2 of rank 2. Every relabelling $\alpha : 2 \to 2$ is permitted for these yielding 1 and 2, respectively.

  For every $n \in \mathbf{N}$, $n \geq 3$, there is a unique constant n. For n, every relabelling $\alpha : n \to n$ is permitted yielding n again.

It is clear that relabellings with the identity morphisms have no effect and the replacement rules are "transitive", that is, Gr is indeed a functor.

In the sequel, we write $A_{\pi,\xi}$, $B_{\pi,\xi}$, $A_{\pi',\xi'}$, and $B_{\pi',\xi'}$ for the new sets A, B, A', and B' introduced above.

As in Section 4, consider again a family $\Psi = \{\psi_A : A \to [||A||] \mid A \in U\}$ of bijections and the functor $\mathbf{\Psi} : \mathbf{U} \to \mathbf{N}$ determined by Ψ. We establish a morphism

$$\langle \mathbf{\Psi}, \omega \rangle : \langle \mathbf{U}, GR \rangle \to \langle \mathbf{N}, Gr \rangle$$

in $\mathbf{ISig}$ by defining the natural transformation $\omega : GR \to \mathbf{ISIG}_{\mathbf{\Psi}} Gr$ as follows

- For each triple $r = \langle A, B; A \ominus B \rangle \in U^3$, the mapping $\omega r : GR(r) \to (\mathbf{ISIG}_{\mathbf{\Psi}} Gr)(r)$ is defined by

$$\omega r([A, B]) = [\pi, \xi]$$

  where

$$\pi = \psi_A^{-1} \circ \mid_{A \cap B} \circ \psi_B$$

  is the (middle) restriction of $\psi_A^{-1} \circ \psi_B$ to $A \cap B$ and

$$\xi = (\psi_A^{-1}[A, B]\psi_B^{-1}) \circ \psi_{A \ominus B}.$$

- For each $A \in U$, let $\omega\langle ; A \rangle(n_A) = n$.

For other ranks r, ωr is the unique morphism $\{*\} \mapsto (\mathbf{ISIG}_{\mathbf{\Psi}} GR)(r)$ in $\mathbf{Set}^*$.

Lemma 5.4 *Let $r = \langle A, B; A \ominus B\rangle \in U^3$ and $r' = \langle A', B'; A' \ominus B'\rangle \in U^3$ be such that*

$$(\omega r)([A,B]) = (\omega r')([A',B']) = [\pi, \xi]$$

and let $\alpha : A \to A'$, $\beta : B \to B'$, and $\gamma : A \ominus B \to A' \ominus B'$ be the morphisms defined by

$$\psi_A = \alpha \circ \psi_{A'} \quad \psi_B = \beta \circ \psi_{B'} \quad \text{and} \quad \psi_{A\ominus B} = \gamma \circ \psi_{A'\ominus B'},$$

that is, $\mathbf{\Psi}\alpha$, $\mathbf{\Psi}\beta$ and $\mathbf{\Psi}\gamma$ are the identity morphisms. Then $\gamma = \alpha \ominus \beta$.

Proof: By the definition of π and the assumption one has

$$\pi = \psi_A^{-1} \circ \ |_{A\cap B} \circ \psi_B = \psi_{A'}^{-1} \circ \ |_{A'\cap B'} \circ \psi_{B'}.$$

Therefore, α and β are compatible. By the definition of ξ and the assumption one has

$$\begin{aligned}\xi &= (\psi_A^{-1}[A,B]\psi_B^{-1}) \circ \psi_{A\ominus B} \\ &= (\psi_A^{-1}[A,B]\psi_B^{-1}) \circ \gamma \circ \psi_{A'\ominus B'}\end{aligned}$$

and also, using Proposition 4.3,

$$\begin{aligned}\xi &= (\psi_{A'}^{-1}[A',B']\psi_{B'}^{-1}) \circ \psi_{A'\ominus B'} \\ &= Ord(\alpha \ominus \beta)(\psi_A^{-1}[A,B]\psi_B^{-1}) \circ \psi_{A'\ominus B'} \\ &= (\psi_A^{-1}[A,B]\psi_B^{-1}) \circ (\alpha \ominus \beta) \circ \psi_{A'\ominus B'}.\end{aligned}$$

This implies $\gamma = \alpha \ominus \beta$. $\square$

Theorem 5.5 *Let $r = \langle A, B; A \ominus B\rangle \in U^3$ and $r' = \langle A', B'; A' \ominus B'\rangle \in U^3$ be arbitrary ranks, and let $\langle \alpha, \beta; \gamma\rangle : r \to r'$ be a rank relabelling in GR. For any family $\Psi = \{\psi_A : A \to [\|A\|] \mid A \in U\}$ of bijections one has*

$$\langle \alpha, \beta; \gamma\rangle : [A,B] \mapsto [A',B']$$

in GR if and only if

$$\langle \mathbf{\Psi}\alpha, \mathbf{\Psi}\beta; \mathbf{\Psi}\gamma\rangle : \omega r([A,B]) \mapsto \omega r'([A',B'])$$

in Gr.

Proof: Let $\omega r([A,B]) = [\pi,\xi]$ and $\omega r'([A',B']) = [\pi',\xi']$. Let $\|A\| = n$, $\|B\| = m$, and $\|A \cap B\| = q$. We extend the category $\mathbf{U}$ by adding new objects $A_{\pi,\xi}$, $B_{\pi,\xi}$, and $A_{\pi,\xi}\ominus B_{\pi,\xi}$, which are isomorphic copies of the sets $[q]+[n-q]$, $[q]+[m-q]$ and $[n-q]+[m-q]$ as constructed above, respectively, and the necessary morphisms. Let $\mathbf{U}'$ denote the extended category. Note that $\|A_{\pi,\xi}\| = \|A\|$, $\|B_{\pi,\xi}\| = \|B\|$, $\|A_{\pi,\xi} \ominus B_{\pi,\xi}\| = \|A \ominus B\|$. We also extend $\mathbf{\Psi}$ to $\mathbf{U}'$ by letting

$$\psi_{A_{\pi,\xi}} = \varphi_{A_{\pi,\xi}}^{-1}, \quad \psi_{B_{\pi,\xi}} = \varphi_{B_{\pi,\xi}}^{-1}, \quad \text{and} \quad \psi_{A_{\pi,\xi}\ominus B_{\pi,\xi}} = \varphi_{A_{\pi,\xi}\ominus B_{\pi,\xi}}^{-1}.$$

Lemma 5.3 shows that

$$\omega\langle A_{\pi,\xi}, B_{\pi,\xi}; A_{\pi,\xi} \ominus B_{\pi,\xi}\rangle([A_{\pi,\xi}, B_{\pi,\xi}]) = [\pi, \xi].$$

Let $\alpha_{\pi,\xi} : A_{\pi,\xi} \to A$, $\beta_{\pi,\xi} : B_{\pi,\xi} \to B$ and $\gamma_{\pi,\xi} : A_{\pi,\xi} \ominus B_{\pi,\xi} \to A \ominus B$ be the bijections for which

$$\Psi\alpha_{\pi,\xi} = id_n, \quad \Psi\beta_{\pi,\xi} = id_m, \quad \text{and} \quad \Psi\gamma_{\pi,\xi} = id_{n+m-2q}.$$

By Lemma 5.4, $\gamma_{\pi,\xi} = \alpha_{\pi,\xi} \ominus \beta_{\pi,\xi}$.

In a similar fashion we now extend $\mathbf{U}'$ by adding further new objects $A_{\pi',\xi'}$, $B_{\pi',\xi'}$, and $A_{\pi',\xi'} \ominus B_{\pi',\xi'}$ corresponding to $[\pi', \xi']$ and by extending GR and Ψ.

It follows that

$$\langle \alpha, \beta; \gamma\rangle$$

is allowable for $[A, B]$ if and only if

$$\langle \alpha^*, \beta^*; \gamma^*\rangle = \langle \alpha_{\pi,\xi} \circ \alpha \circ \alpha^{-1}_{\pi',\xi'}, \beta_{\pi,\xi} \circ \beta \circ \beta^{-1}_{\pi',\xi'}; \gamma_{\pi,\xi} \circ \gamma \circ \gamma^{-1}_{\pi',\xi'}\rangle$$

is allowable for $[A_{\pi,\xi}, B_{\pi,\xi}]$. By definition, $\langle \alpha^*, \beta^*; \gamma^*\rangle$ is allowable for $[A_{\pi,\xi}, B_{\pi,\xi}]$ if and only if

$$\langle \Psi\alpha^*, \Psi\beta^*; \Psi\gamma^*\rangle : [\pi, \xi] \mapsto [\pi', \xi']$$

in Gr. But,

$$\Psi\alpha^* = \Psi\alpha_{\pi,\xi} \circ \Psi\alpha \circ \Psi\alpha^{-1}_{\pi',\xi'} = \Psi\alpha$$

and, similarly, $\Psi\beta^* = \Psi\beta$ and $\Psi\gamma^* = \Psi\gamma$. □

Corollary 5.6 $\omega : GR \to \mathbf{ISIG}_\Psi Gr$ *is a non-degenerate natural embedding.*

Proof: ω is natural by Theorem 5.5. It is also an embedding, because $GR(r)$ always contains at most one operation symbol, except in the trivial case $r = \langle ; A\rangle$ when A is a two element set. □

6. Indexed Algebras and M

In this chapter, we perform the final step of our construction. We consider GR-algebras—U-indexed algebras—and obtain a representation theorem which states that every GR-algebra is naturally isomorphic with a GR-algebra constructed from a Gr-algebra by the functor $\mathbf{Con}^{-1}$. Again, the algebra M of undirected multigraphs is a special case.

Definition 6.1 Let Σ be an $\mathbf{S}$-indexed algebraic signature. A Σ-*algebra* is an $\mathbf{S}$-indexed family of sets $A : \mathbf{S} \to \mathbf{Set}$, called the *carrier* of the algebra, which is equipped with an operation $\Diamond_A : As_1 \times \cdots \times As_n \to As_{n+1}$ for each operation symbol $\Diamond$ in $\Sigma\langle s_1, \ldots, s_n; s_{n+1}\rangle$ satisfying the following condition. For any two objects $r = \langle s_1, \ldots, s_n; s_{n+1}\rangle$, $r' = \langle s'_1, \ldots, s'_n; s'_{n+1}\rangle$ and any morphism $\alpha = \langle \alpha_1, \ldots, \alpha_n; \alpha_{n+1}\rangle$ in $\mathbf{S}^+$, if $\Diamond$ and $\Diamond'$ are operation symbols in Σr and $\Sigma r'$, respectively, and

$$\alpha : \Diamond \mapsto \Diamond'$$

in Σ then the diagram

$$\begin{tikzcd}
As_1 \times \cdots \times As_n \arrow[rr, "\Diamond"] \arrow[d, "A\alpha_1 \times \cdots \times A\alpha_n"'] & & As_{n+1} \arrow[d, "A\alpha_{n+1}"] \\
As'_1 \times \cdots \times As'_n \arrow[rr, "\Diamond'"'] & & As'_{n+1}
\end{tikzcd}$$

commutes.

Definition 6.2 Let A and B be Σ-algebras. A Σ-*homomorphism* from A into B is a natural transformation $h : A \to B$ which, at the same time, is a homomorphism of $|\mathbf{S}|$-sorted algebras.

Let $\mathbf{IALG}(\Sigma)$ denote the category of Σ-algebras. Every morphism $\langle \mathbf{\Lambda}, \omega\rangle : \langle \mathbf{S}, \Sigma\rangle \to \langle \mathbf{S}', \Sigma'\rangle$ in $\mathbf{ISig}$ defines a functor

$$\_|_{\langle \mathbf{\Lambda}, \omega\rangle} : \mathbf{IALG}(\Sigma') \to \mathbf{IALG}(\Sigma)$$

as follows.

- *On objects:* Let A' be a Σ'-algebra. We define the Σ-algebra $A = A' \,|_{\langle \mathbf{\Lambda}, \omega\rangle}$.
  - For all objects $s \in |\mathbf{S}|$, one has $As = A'(\mathbf{\Lambda} s)$.
  - For all morphisms α in $\mathbf{S}$, one has $A\alpha = A'(\mathbf{\Lambda}\alpha)$.
  - For all $r = \langle s_1, \ldots, s_n; s_{n+1}\rangle \in |\mathbf{S}^+|$ and $\Diamond \in \Sigma r \setminus \{*\}$, one has $\Diamond_A = (\omega\Diamond)_{A'}$ (note that ω is non-degenerate).
- *On morphisms:* $\_|_{\langle \mathbf{\Lambda}, \omega\rangle}$ acts as in $\mathbf{ISET}$, forgetting the algebraic structure (see also [Bu1], [Bu2]).

Thus we have defined the indexed category

$$\mathbf{IALG} : \mathbf{ISig}^{\mathrm{op}} \to \mathbf{Cat}$$

of indexed algebras. Flattening $\mathbf{IALG}$ yields the category $\mathbf{Flat}(\mathbf{IALG})$ whose objects are pairs $\langle \Sigma, A\rangle$, where A is a Σ-algebra. A morphism from $\langle \Sigma, A\rangle$ to $\langle \Sigma', A'\rangle$ is a signature morphism $\langle \mathbf{\Lambda}, \omega\rangle : \langle \mathbf{S}, \Sigma\rangle \to \langle \mathbf{S}', \Sigma'\rangle$ together with a Σ-homomorphism $h : A \to A' \,|_{\langle \mathbf{\Lambda}, \omega\rangle}$.

Remark 6.3 *The algebra M of undirected multigraphs is a GR-algebra. Its carrier is the functor $M : \mathbf{U} \to \mathbf{Set}$. The operations satisfy the compatibility conditions of Definition 6.1.*

Remark 6.4 *Let GR' denote the restriction of the signature GR to its binary operation symbols. The functor $\mathrm{Ord} : \mathbf{U} \to \mathbf{Set}$ equipped with the operations $[A, B]$ is a GR'-algebra by Proposition 4.3.*

In the rest of the paper we show that every GR-algebra can be contracted to a Gr-algebra in a natural way.

Definition 6.5 Let $F : \mathbf{U} \to \mathbf{Set}$ be an arbitrary GR-algebra. We introduce the Gr-operations on $\mathbf{Con}F : \mathbf{N} \to \mathbf{Set}$ in the following way.

- Consider $a \in (\mathbf{Con}F)_n$, $b \in (\mathbf{Con}F)_m$ and $[\pi, \xi] \in Gr\langle n, m; n + m - 2q\rangle$. Let $\varphi_A : [n] \to A$ and $\varphi_B : [m] \to B$ be bijections satisfying $\pi = \varphi_A \circ |_{A \cap B} \circ \varphi_B^{-1}$. Define $a[\pi, \xi]b$ by

$$a[\pi, \xi]b = F((\varphi_A[A, B]\varphi_B)^{-1} \circ \xi)((F\varphi_A)(a)[A, B](F\varphi_B)(b)).$$

- Any constant of rank n in $\mathbf{Con}F$ is the same as the corresponding one of rank $[n]$ in F.

Theorem 6.6 *In Definition 6.5, $a[\pi, \xi]b$ does not depend on the choice of φ_A and φ_B.*

Rather than prove this claim directly, we formulate an equivalent statement and prove that one as a consequence of Lemma 5.4. To do so, we introduce the following alternative definition of $a[\pi, \xi]b$.

Definition 6.7 Let $\Psi = \{\psi_A : A \to [||A||] \mid A \in U\}$ be an arbitrary family of bijections, and suppose that A, B, and $A \ominus B$ are such that $\omega\langle A, B; A \ominus B\rangle = [\pi, \xi]$ where $\omega : GR \to \mathbf{ISIG}_{\Psi} Gr$ is the natural embedding justified by Theorem 5.5. For $a \in F[n]$ and $b \in F[m]$, define $a[\pi, \xi]b$ by

$$a[\pi, \xi]b = (F\psi_{A \ominus B})((F\psi_A^{-1})(a)[A, B](F\psi_B^{-1})(b)).$$

Theorem 6.8 *In Definition 6.7, $a[\pi, \xi]b$ does not depend on the choice of the objects A and B.*

Proof: Suppose that also $\omega\langle A', B'; A' \ominus B'\rangle = [\pi, \xi]$. Let $\alpha : A \to A'$, $\beta : B \to B'$, and $\gamma : A \ominus B \to A' \ominus B'$ be those bijections for which $\mathbf{\Psi}\alpha$, $\mathbf{\Psi}\beta$, and $\mathbf{\Psi}\gamma$ are the identities. By Lemma 5.4, $\gamma = \alpha \ominus \beta$. Since F is a GR-algebra,

$$(F\psi_{A'}^{-1})(a)[A', B'](F\psi_{B'}^{-1})(b) = F(\alpha \ominus \beta)((F\psi_A^{-1})(a)[A, B](F\psi_B^{-1})(b)).$$

On the other hand,

$$F\psi_{A \ominus B} = F\gamma \circ F\psi_{A' \ominus B'} = F(\alpha \ominus \beta) \circ F\psi_{A' \ominus B'}.$$

Thus,

$$(F\psi_{A \ominus B})((F\psi_A^{-1})(a)[A, B](F\psi_B^{-1})(b)) = (F\psi_{A' \ominus B'})((F\psi_{A'}^{-1})(a)[A', B'](F\psi_{B'}^{-1})(b)).$$

□

Corollary 6.9 *The definitions of $a[\pi,\xi]b$ in 6.5 and 6.7are equivalent.*

Proof: Choose $\psi_A = \varphi_A^{-1}$, $\psi_B = \varphi_B^{-1}$ and $\psi_{A\ominus B} = \varphi_{A\ominus B}^{-1}$ in Definition 6.5. $\square$

Corollary 6.10 **Con**F *is a Gr-algebra, and* **Con** : **IALG**$(GR) \rightarrow$ **IALG**(Gr) *is a functor.*

Proof: The first statement follows immediately from Corollary 6.9 and Theorem 5.5. The second statement is then obvious. $\square$

Note that the main point of the above three simple proofs is that we derive a non-obvious semantic feature of Gr-algebras from the corresponding obvious semantic feature of GR-algebras and a purely syntactic connection between their signatures.

Corollary 6.11 **Con**M *is a Gr-algebra.*

Let $n, m, q \in \mathbf{N}$, $q \leq \min(n,m)$, and consider the sets $A_{\pi,\xi} = [q] + [n-q]$, $B_{\pi,\xi} = [q] + [m-q]$ and the mappings $\varphi_{A_{\pi,\xi}} : [n] \rightarrow A_{\pi,\xi}$, $\varphi_{B_{\pi,\xi}} : [n] \rightarrow B_{\pi,\xi}$ as defined in Section 5. Identifying $[q]+[n-q]$ with $[n]$ and $[q]+[m-q]$ with $[m]$ yields two permutations $\bar{\varphi}_{A_{\pi,\xi}} : [n] \rightarrow [n]$ and $\bar{\varphi}_{B_{\pi,\xi}} : [m] \rightarrow [m]$.

Corollary 6.12 *In $G =$ **Con**F one has*

$$a[\pi,\xi]b = (G\xi)((G\bar{\varphi}_{A_{\pi,\xi}})(a)[id_q, id_{n+m-2q}](G\bar{\varphi}_{B_{\pi,\xi}})(b)).$$

Proof: One observes that the replacement

$$[\pi,\xi] \mapsto [id_q, id_{n+m-2q}]$$

is permitted by

$$\langle \bar{\varphi}_{A_{\pi,\xi}}, \bar{\varphi}_{B_{\pi,\xi}}; \xi^{-1} \rangle.$$

$\square$

Corollary 6.11 allows us to characterize $[\pi,\xi]$ as an operation which can be derived from the "basic" one $[id_q, id_{n+m-2q}]$, just as in the case of undirected multigraphs.

Theorem 6.13 *For every family $\Psi = \{\psi_A : A \rightarrow [\|A\|] \mid A \in U\}$ of bijections one has*

$$id_{\mathbf{IALG}(GR)} \cong \mathbf{Con} \circ \mathbf{IALG}_{\langle \Psi,\omega \rangle}$$

and

$$id_{\mathbf{IALG}(Gr)} \cong \mathbf{IALG}_{\langle \Psi,\omega \rangle} \circ \mathbf{Con}$$

using appropriate natural isomorphisms.

Proof: One verifies that the natural isomorphisms μ and ν defined in the proof of Theorem 4.2 are appropriate. One needs to show that for every GR-algebra F, μF is a GR-homomorphism; this is immediate by 6.7. Using the proof of Theorem 4.2, the corresponding statement for ν is then a consequence. $\square$

Finally, as in Section 4, we can eliminate the index functor $\mathbf{\Psi}$ and replace the functor $\mathbf{IALG}_{\langle\mathbf{\Psi},\omega\rangle}$ by a "canonical inverse" $\mathbf{Con}^{-1}$ of $\mathbf{Con}$. For this purpose, we again consider the functor $\mathbf{Con}^{-1} : \mathbf{ISET(N)} \to \mathbf{ISET(U)}$. It can be turned into a functor $\mathbf{IALG}(Gr) \to \mathbf{IALG}(GR)$ as follows. Let G be a Gr-algebra. Then we define a GR-algebra on $\mathbf{Con}^{-1}G$:

- Consider $A, B \in \mathbf{U}$ with $\|A\| = n$, $\|B\| = m$, and $\|A \cap B\| = q$. We need to define

$$[A, B] : (\mathbf{Con}^{-1}G)A \times (\mathbf{Con}^{-1}G)B \to (\mathbf{Con}^{-1}G)(A \ominus B).$$

  Given $a \in G_n$, $b \in G_m$, $\varphi_A : [n] \to A$, $\varphi_B : [m] \to B$, let

$$(a, \varphi_A)^{\sim}[A, B](b, \varphi_B)^{\sim} = (a[\pi, id_{n+m-2q}]b, (\varphi_A[A, B]\varphi_B))^{\sim}$$

  where

$$\pi = \varphi_A \circ \mid_{A \cap B} \circ \varphi_B^{-1}$$

  and where $^{\sim}$ denotes equivalence classes.
- For any constant n in G and appropriate $A \in U$ and φ_A, let n_A be the equivalence class $(n, \varphi_A)^{\sim}$.

Of course, one has to verify, that these definitions are correct. For $[A, B]$, and using the above notation, one obtains

$$(a, \varphi_A)[A, B](b, \varphi_B) \sim (a', \varphi'_A)[A, B](b', \varphi'_B)$$

if

$$(a, \varphi_A) \sim (a', \varphi'_A) \quad \text{and} \quad (b, \varphi_B) \sim (b', \varphi'_B)$$

using 6.5. For n_A, note that $(n, \varphi_A) \sim (n, \varphi'_A)$ for all φ_A and φ'_A.

In this way, the natural isomorphism $\mathbf{ISET}_{\mathbf{\Psi}}G \cong \mathbf{Con}^{-1}G$ of indexed families of sets induces a natural isomorphism $\mathbf{IALG}_{\langle\mathbf{\Psi},\omega\rangle}G \cong \mathbf{Con}^{-1}G$ of indexed algebras. This shows that $\mathbf{IALG}_{\langle\mathbf{\Psi},\omega\rangle}$ can be replaced by $\mathbf{Con}^{-1}$ in Theorem 6.13. Denoting the algebra $\mathbf{Con}^{-1}G$ by $G[Ord]$, we obtain a representation theorem for $\mathbf{U}$-indexed algebras which corresponds to Theorem 4.4, that is the representation theorem for $\mathbf{U}$-indexed families of sets.

Theorem 6.14 *Every GR-algebra F is naturally isomorphic with $G[Ord]$ for a suitable Gr-algebra G.*

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