MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 **MATH 1000-5** **March 24, 2003**

1. Find the derivative. Simplify where possible.

(a) $y = x^5(x^3-2)^4$ (Factorize the answer.)

By Product Rule $y' = 5x^4(x^3-2)^4 + x^5\,[(x^3-2)^4]'$

By Chain Rule $[(x^3-2)^4]' = 4\,(x^3-2)^3 \cdot (3x^2) \;=\; 12\,(x^3-2)^3x^2.$

Finally, $y' \;=\; 5x^4(x^3-2)^4 + x^5 \cdot 12\,(x^3-2)^3x^2$

$$= \; x^4\,(x^3-2)^3\,(5(x^3-2)+12x^3) \;=\; x^4\,(x^3-2)^3\,(17x^3-10).$$

(b) $y = \ln|6x| + 6^x + \log_6 x$

Answer: $$y' \;=\; \frac{1}{x} + 6^x \ln 6 + \frac{1}{x\ln 6}.$$

(c) $y = 2\cos^3\sqrt{1+x^2}$

$$y = f(g(x)), \qquad f(t) = 2\cos^3 t, \qquad g(x) = \sqrt{1+x^2}.$$

$$f'(t) = -6\cos^2 t\,\sin t,$$

$$g'(x) = [(1+x^2)^{1/2}]' = \frac{1}{2}(1+x^2)^{-1/2}\cdot(2x) = \frac{x}{\sqrt{1+x^2}}.$$

Finally, $$y' \;=\; -6\,\cos^2\sqrt{1+x^2}\,\sin\sqrt{1+x^2}\,\frac{x}{\sqrt{1+x^2}}.$$

(d) $y = \ln(\tan x) - \tan(\ln x)$ Let $t = \tan x$, then

$$(\ln(\tan x))' = (\ln t)'\,(\tan x)' \;=\; \frac{1}{\tan x}\cdot(\sec^2 x) \;=\; \frac{\cos x}{\sin x}\,\frac{1}{\cos^2 x} \;=\; \frac{1}{\sin x\cos x},$$

$$(\tan\ln x)' = (\sec^2\ln x)\cdot\frac{1}{x}.$$

Finally, $$y' \;=\; \frac{1}{\sin x\cos x} - \frac{\sec^2\ln x}{x}.$$

(e) $3x^2\,y^4 = 2y^5 + 4x^3$

Implicit differentiation: $$3\left(2x\,y^4 + 4x^2\,y^3\,\frac{dy}{dx}\right) = 10y^4\,\frac{dy}{dx} + 12x^2,$$

$$\frac{dy}{dx}\left(12x^2y^3 - 10y^4\right) \;=\; 12x^2 - 6xy^4,$$

Finally, $$\frac{dy}{dx} \;=\; \frac{12x^2-6xy^4}{12x^2y^3-10y^4} \;=\; \frac{3x\,(2x-y^4)}{y^3\,(6x^2-5y)}.$$

2. Use logarithmic differentiation to find the derivative.

(b) $y = \dfrac{2^x\,(x^4+1)^5}{\sqrt{x+x^2}}$

$$\ln y = x\ln 2 + 5\ln(x^4+1) - \frac{1}{2}\ln(x+x^2),$$

$$\frac{y'}{y} = \ln 2 + \frac{5\cdot 4x^3}{x^4+1} - \frac{1+2x}{2(x+x^2)}$$

Finally, $$y' = \frac{2^x\,(x^4+1)^5}{\sqrt{x+x^2}}\left(\ln 2 + \frac{20x^3}{x^4+1} - \frac{1+2x}{2x(1+x)}\right).$$

3. The length of a rectangle is increasing at a rate of 3 cm/min while the width is decreasing at a rate of 4 cm/min.

(a) Find the rate at which the area of the rectangle is changing when the length is 30 cm and the width is 70 cm.

Let l be the length, w the width, A the area. Then $\frac{dl}{dt} = 3$ cm/min, $\frac{dw}{dt} = -4$;cm/min. The area is $A = l\cdot w$. Hence,

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt} = 30\cdot(-4) + 70\cdot 3 = -120 + 210 = \ 90\ \ \text{cm}^2/\text{min}.$$

(b) Find the rate at which the length of the diagonal is changing when the length is 30 cm and the width is 40 cm.

(a) Let x be the diagonal. Then $x^2 = l^2 + w^2$. Hence,

$$2x\frac{dx}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt},$$

$$\frac{dx}{dt} = \frac{1}{x}\left(l\frac{dl}{dt} + w\frac{dw}{dt}\right)$$

$$= \frac{1}{\sqrt{30^2+40^2}}\,(30\cdot 3 - 40\cdot 4) = \frac{90-160}{50} = \frac{-7}{5}\ \text{cm/min}.$$

4. Given $f(x) = x^4 - 2x^2$, use differentiation to determine the intervals over which the function is increasing, decreasing, concave up or concave down. Find the exact values of both coordinates of all extreme points, inflection points and intercepts.

$$f(x) = x^4 - 2x^2 = x^2\,(x^2-2),$$
$$f'(x) = 4x^3 - 4x = 4x\,(x^2-1),$$
$$f''(x) = 12x^2 - 4 = 4(3x^2-1).$$

x-intercepts are found from the eqn. $f(x) = 0$: $x = 0,\ x = \pm\sqrt{2}$.
The x-intercepts are $(-\sqrt{2},0)$, $(0,0)$, $(\sqrt{2},0)$. The y-intercept is $(0,0)$.

Relative extrema are found from the eqn. $f'(x) = 0$: $x = 0,\ x = \pm 1$.
Intervals of monotonicity: $I = (-\infty,-1)$, $II = (-1,0)$, $III = (0,1)$. $IV = (1,\infty)$. Signs: $f' < 0$ in I and III; $f' > 0$ in II and IV. Therefore the critical points are:

Relative minimum at $x = -1$, $y = (-1)^4 - 2(-1)^2 = -1$.

Relative maximum at $x = 0$, $y = 0$.

Again relative minimum at $x = 1$, $y = -1$

$f(0) = 1, \ f(1) = 0, \ f(2) = 1$

$f'(1) = 0$

$f''(x) < 0$ when $x < -1$ and when $x > 2$

$f''(x) > 0$ when $-1 < x < 2$

$\lim_{x \to -1^-} f(x) = -\infty, \quad \lim_{x \to -1^+} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -1, \quad \lim_{x \to \infty} f(x) = 2.$

Solution. The information provided should be interpreted as follows:

1. Put the points $(0, 1)$, $(1, 0)$ and $(2, 1)$ on the graph.
2. Draw the horizontal asymptotes: $y = -1$ to the left and $y = 2$ to the right. Draw the vertical asymptote $x = -1$.
3. The function has horizontal tangent at $x = 1$, which corresponds to one of the given points: $(1, 0)$. No other critical points exist.
4. The function is concave down in $(-\infty, -1)$. Then it is concave up in $(-1, 2)$, and again concave down in $(2, \infty)$.
5. Sketch the graph:

As x goes to $-\infty$, the graph approaches the asymptote $y = -1$ from below and it is concave down. The function decreases when $x < -1$.

As x approaches -1 from the left, the graph approaches the negative direction $y \to -\infty$ of the vertical asymptote $x = -1$.

As x approaches -1 from the right, the graph approaches the positive direction $y \to \infty$ of the vertical asymptote $x = -1$.

The graph goes down as x changes from -1 to 1. It has relative minimum at $(1, 0)$ and then increases, approaching the horizontal asymptote $y = 2$ from below as x goes to $+\infty$. The inflection point is $(2, 1)$. To the right of it the function is concave down, and to the left concave up.