MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 (Sample) Version B MATH 1000-5 Solutions

1. (a) $y = x^3(3x^4+1)^5$ (Factorize the answer.)

By Product Rule $y' = 3x^2(3x^4+1)^5 + x^3\,[(3x^4+1)^5]'$

By Chain Rule $[(3x^4+1)^5]' = 5\,(3x^4+1)^4 \cdot (3\cdot 4x^3+0) \;=\; 60\,(3x^4+1)^4 x^3.$

Finally, $y' \;=\; 3x^2(3x^4+1)^5 + x^3 \cdot 60\,(3x^4+1)^4 x^3$

$$= \; 3x^2\,(3x^4+1)^4\,(3x^4+1+20x^4) \;=\; 3x^2\,(3x^4+1)^4\,(23x^4+1).$$

(b) $y = \log_5 x + \ln|5x| + 5^x$

$$(\log_5 x) = \left(\frac{\ln x}{\ln 5}\right)' = \frac{1}{x\ln 5}$$

$$(\ln|5x|)' = (\ln 5 + \ln|x|)' = 0 + \frac{1}{x}, \qquad (5^x)' = 5^x \ln 5$$

Finally, $$y' \;=\; \frac{1}{x\ln 5} \;+\; \frac{1}{x} \;+\; 5^x \ln 5.$$

(c) $y = 6\sin^3\sqrt{1-x^2}$

$$y = f(g(x)), \qquad f(t) = 6\sin^3 t, \qquad g(x) = \sqrt{1-x^2}.$$

$$f'(t) = 18\sin^2 t\,\cos t,$$

$$g'(x) = [(1-x^2)^{1/2}]' = \frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{1-x^2}}.$$

Finally, $$y' \;=\; \frac{-18\,x\,\sin^2\sqrt{1-x^2}\,\cos\sqrt{1-x^2}}{\sqrt{1-x^2}}.$$

(d) $y = \cot(\ln x) + \ln(\sec x)$ Let $t = \sec x$, then

$$(\ln(\sec x))' = (\ln t)'\,(\sec x)' \;=\; \frac{1}{\sec x}\cdot(\sec x\,\tan x) = \tan x,$$

$$(\cot\ln x)' = (-\csc^2\ln x)\cdot\frac{1}{x}$$

Finally, $$y' \;=\; \frac{-\csc^2\ln x}{x} + \tan x.$$

(e) $5x^4\,y^3 = 4y^3 + 3x^4$

Implicit differentiation: $\quad 5\left(4x^3\,y^3 \;+\; 3x^4\,y^2\,\dfrac{dy}{dx}\right) \;=\; 12y^2\,\dfrac{dy}{dx} \;+\; 12x^3,$

$\dfrac{dy}{dx}\,\left(15x^4y^2 - 12y^2\right) \;=\; 12x^3 - 20x^3y^3,$

Finally, $\quad \dfrac{dy}{dx} \;=\; \dfrac{12x^3 - 20x^3y^3}{15x^4y^2 - 12y^2} \;=\; \dfrac{4x^3\,(3-5y^3)}{3y^2\,(5x^4-4)}.$

2. (Logarithmic differentiation)

(a) $\quad y = \dfrac{\sqrt{x^4 - x}}{7^x(1+x^2)^6}$

$$\ln y = \frac{1}{2}\ln(x^4 - x) - x\ln 7 - 6\ln(1+x^2),$$

$$\frac{y'}{y} \;=\; \frac{4x^3-1}{2(x^4-x)} \;-\; \ln 7 \;-\; \frac{6\cdot 2x}{1+x^2}$$

Finally, $\quad y' \;=\; \dfrac{\sqrt{x^4-x}}{7^x(1+x^2)^6}\,\left(\dfrac{4x^3-1}{2(x^4-x)} \;-\; \ln 7 \;-\; \dfrac{12x}{1+x^2}\right).$

(b) $\quad y = (3x+2)^{\sqrt{5x-4}}$

$$\ln y = \;\sqrt{5x-4}\;\ln(3x+2),$$

$$\frac{y'}{y} \;=\; \sqrt{5x-4}\,\frac{3}{3x+2} \;+\; \frac{1}{2}\,\frac{5}{\sqrt{5x-4}}\;\ln(3x+2)$$

$$y' \;=\; (3x+2)^{\sqrt{5x-4}}\left(\frac{3\sqrt{5x-4}}{3x+2} \;+\; \frac{5\,\ln(3x+2)}{2\sqrt{5x-4.}}\right).$$

3. Given $f(x) = 5x^2 - 5x^4$, use differentiation to determine the intervals over which the function is increasing, decreasing, concave up or concave down. Find the exact values of both coordinates of all extreme points, inflection points and intercepts.

Solution:

$$f(x) = 5x^2 - 5x^4 = 5x^2\,(1-x)(1+x),$$
$$f'(x) = 10x - 20x^3 = 10x\,(1-2x^2),$$
$$f''(x) = 10 - 60x^2 = 10(1-6x^2).$$

x-intercepts are found from the eqn. $f(x) = 0$: $\;x = 0,\ x = \pm 1$.

The x-intercepts are $(-1,0)$, $(0,0)$, $(1,0)$. The y-intercept is $(0,0)$.

Relative extrema are found from the eqn. $f'(x) = 0$: $\;x = 0,\ x = \pm\sqrt{\tfrac{1}{2}}$.

Intervals of monotonicity: $I = (-\infty, -\sqrt{\frac{1}{2}})$, $II = (-\sqrt{\frac{1}{2}}, 0)$, $III = (0, \sqrt{\frac{1}{2}})$. $IV = (\sqrt{\frac{1}{2}}, \infty)$. Signs: $f' > 0$ in I and III; $f' < 0$ in II and IV. Therefore the critical points are:

Relative maximum at $x = -\sqrt{\frac{1}{2}}$, $y = 5((\sqrt{\frac{1}{2}})^2 - (\sqrt{\frac{1}{2}})^4) = 5(1/2 - 1/4) = 5/4$.

Relative minimum at $x = 0$, $y = 0$.

Again relative maximum at $x = \sqrt{\frac{1}{2}}$, $y = 5/4$.

Inflection points are found from the eqn. $f''(x) = 0$: $x = \pm\frac{1}{\sqrt{6}}$.

Intervals of concativity: $I = (-\infty, -\frac{1}{\sqrt{6}})$, $II = (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$, $III = (\frac{1}{\sqrt{6}}, \infty)$. Signs: $f'' < 0$ in I and III (Concave down); $f' > 0$ in II (Concave up). When $x = \pm 1/\sqrt{6}$, $y = 5(1/6 - 1/36) = 5\frac{6-1}{36} = \frac{25}{36}$. The inflection points are $(-\frac{1}{\sqrt{6}}, \frac{25}{36})$ and $(\frac{1}{\sqrt{6}}, \frac{25}{36})$.

4. The length of a rectangle is decreasing at a rate of 4 cm/sec while the width is increasing at a rate of 5 cm/sec.

(a) Find the rate at which the area of the rectangle is changing when the length is 30 cm and the width is 20 cm.

Let l be the length, w the width, A the area. The data given: $\frac{dl}{dt} = -4$ cm/sec, $\frac{dw}{dt} = 5$ cm/sec. The area is $A = l \cdot w$. Hence,

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt} = 30 \cdot 5 + 20 \cdot (-4) = 150 - 80 = \; 70 \;\; \text{cm}^2/\text{sec}.$$

(b) Find the rate at which the length of the diagonal is changing when the length is 12 cm and the width is 5 cm.

(a) Let x be the diagonal. Then $x^2 = l^2 + w^2$. Hence,

$$\begin{aligned} 2x\frac{dx}{dt} &= 2l\frac{dl}{dt} + 2w\frac{dw}{dt}, \\ \frac{dx}{dt} &= \frac{1}{x}\left(l\frac{dl}{dt} + w\frac{dw}{dt}\right) \\ &= \frac{1}{\sqrt{12^2 + 5^2}}\,(12 \cdot (-4) + 5 \cdot 5) \;=\; \frac{-48 + 25}{\sqrt{144 + 25}} = \frac{-23}{13} \;\; \text{cm/sec.} \end{aligned}$$

5. Sketch a graph of a function that satisfies all the given conditions. Clearly label any extreme points, inflection points and asymptotes.

Domain $= \{x \mid x \neq 1\}$

$f(0) = 1$, $f(-2) = 1$, $f(-3) = 0$

$f'(-2) = 0$

$f''(x) > 0$ when $x < -3$ and when $x > 1$

$f''(x) > 0$ when $-3 < x < 1$

$\lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = +\infty$
$\lim_{x\to -\infty} f(x) = -1, \quad \lim_{x\to\infty} f(x) = 0.$

Solution. The sketch is not presented here by technical reason. The information provided should be interpreted as follows:

1. Put the points $(0,0)$, $(-2,1)$ and $(-3,0)$ on the graph.
2. Draw the horizontal asymptotes: $y=-1$ to the left and $y=0$ to the right (it coincides with the positive direction of the x-axis). Draw the vertical asymptote $x=1$.
3. The function has the horizontal tangent at $x=-2$, which corresponds to one of the given points: $(-2,1)$. No other critical points exist.
4. The function is concave down up $(-\infty,-3)$. Then it is concave up in $(-3,1)$, and again concave down in $(1,\infty)$.
5. Sketch the graph:

As x goes to $-\infty$, the graph approaches the horizontal line $y=-1$ from above. The function increases when $x<-2$. It is concave up when $x<-3$, then passes through the inflection point and becomes concave down.
It has relative maximum at $(-2,1)$ and then decreases, approaching the negative direction of the vertical asymptote as $x\to 1^-$.
On the other side of the vertical asymptote the function is decreasing, concave up, and it approaches the the horizontal asymptote $y=0$ from above as x goes to $+\infty$.