MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 **MATH 1000-5** **Solutions**

1. (Limits)
(a) (Divide out. The limit happens to be infinite.)

$$\lim_{x\to 0}\frac{3x-6x^2}{2x^3+x^4}=\lim_{x\to 0}\frac{x(3-6x)}{x^3(2+x)}=\lim_{x\to 0}\frac{3-6x}{x^2(2+x)}$$

As $x\to 0$, we have: $3-6x\to 3$, and $x^2(2+x)\to 0$, while being positive. Therefore, the answer is $+\infty$.

(b) (Rationalize)

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{2x-2}=\lim_{x\to 1}\frac{(x+3)-4}{\sqrt{x+3}+2}\,\frac{1}{2x-2}$$
$$=\lim_{x\to 1}\frac{x-1}{(\sqrt{x+3}+2)\,(x-1)\,2}=\lim_{x\to 1}\frac{1}{2(\sqrt{x+3}+2)}=\frac{1}{2(\sqrt{1+3}+2)}=\frac{1}{8}$$

2. Determine all points at which $f(x)$ is not continuous (explain why), and determine whether each discontinuity is removable. The function is

$$f(x)=\begin{cases}-3|x|, & \text{if } x<-2\\ 6, & \text{if } x=-2\\ \dfrac{x^3+8}{x^2+2x}, & \text{if } x>-2\end{cases}$$

Solution Suspicious points, where the function might be discontinuous, are: 1) points where the formula changes; 2) points where the denominator becomes 0.
1) $x=-2$ (change in the formula). We have:

$$\lim_{x\to -2^-}f(x)=\lim_{x\to -2^-}-3(-x)==-3\cdot 2=-6,$$
$$\lim_{x\to -2^+}f(x)=\lim_{x\to -2^-}\frac{x^3+8}{x^2+2x}=\lim_{x\to -2^+}f(x)=\lim_{x\to -2^-}\frac{(x+2)(x^2-2x+4)}{x(x+2)}$$
$$=\lim_{x\to -2^+}\frac{x^2-2x+4}{x}=\frac{(-2)^2-2\cdot(-2)+4}{-2}=\frac{12}{-2}=-6.$$

Since the right and left limits are equal, the limit $\lim_{x\to -2}f(x)$ exist and equals -6. But $f(-2)\neq -6$. Therefore the function is discontinuous at $x=-2$, but the discontinuity is removable, because there exists the limit.

2) Denominator = 0: solving the equation $x^2 + 2x = 0$, we obtain $x = 0$ or $x = -2$. Case $x = -2$ has already been considered. As $x \to 0$, we have

$$\lim_{x \to 0^{\pm}} f(x) = \lim x \to 0^{\pm} \frac{8}{2 \cdot (\pm 0)} = \pm\infty.$$

Therefore, the function doesn't have a finite value at $x = 0$. It is discontinuous there and the discontinuity is not removable.

3. Find $f'(x)$ using the **definition** of derivative. The function is $f(x) = \frac{2}{x+2}$.

$$f(x+h) = \frac{2}{x+h+2}, \qquad \frac{f(x+h) - f(x)}{h} = \left(\frac{2}{x+h+2} - \frac{2}{x+2}\right)\frac{1}{h}$$

$$= \frac{2(x+2) - 2(x+h+2)}{(x+2)(x+h+2)h} = \frac{-2h}{(x+2)(x+h+2)h} = \frac{-2}{(x+2)(x+2+h)}.$$

Therefore

$$f'(x) = \lim_{h \to 0} \frac{-2}{(x+2)(x+2+h)} = \frac{-2}{(x+2)^2}.$$

4. Find derivatives using differentiation rules.

(a) $f(x) = 2x^2 + \frac{1}{(2x)^2} + e^{\pi}x - \tan\frac{\pi}{360}$

$$f'(x) = 4x + \frac{1}{4}(-2x^-3) + e^{\pi} + 0 = 4x - \frac{1}{2x^3} + e^{\pi}.$$

(b) $f(x) = 3x\sqrt{x} + \frac{x}{\sqrt[4]{x}}$

$$x\sqrt{x} = x^{3/2}, \qquad \frac{x}{\sqrt[4]{x}} = x^{1-1/4} = x^{3/4}$$

$$f'(x) = 3 \cdot \frac{3}{2}x^{\frac{3}{2}-1} + \frac{3}{4}x^{\frac{3}{4}-1} = \frac{9}{2}x^{\frac{1}{2}} + \frac{3}{4}x^{-\frac{1}{4}}$$

(c) $f(x) = -16e^x \ln x$

$$f'(x) = -16\,e^x \ln x - 16\,e^x \frac{1}{x} = -16\,e^x \left(\ln x + \frac{1}{x}\right).$$

(d) $f(x) = \frac{3x^2+1}{x^3-4x}$

$$f'(x) = \frac{6x(x^3-4x) - (3x^2+1)(3x^3-4}{(x^3-4x)^2} = \frac{6x(x^3-4x) - (9x^4+3x^2-12x^2-4)}{(x^3-4x)^2}$$

$$= \frac{6x^4 - 24x^2 - 9x^4 + 9x^2 + 4)}{(x^3-4x)^2} = \frac{-3x^4 - 15x^2 + 4)}{(x^3-4x)^2}.$$

5. Use the Quotient Rule to show that $(x\sec x)' = \sec x\,(1 + x\tan x)$. The key idea is to write $x\sec x$ as $\dfrac{x}{\cos x}$. Then

$$\left(\frac{x}{\cos x}\right)' = \frac{\cos x \,+\, x\,\sin x}{\cos^2 x}.$$

On the other hand, also

$$\sec x\,(1 + x\tan x) = \frac{1}{\cos x}\left(1 + \frac{x\,\sin x}{\cos x}\right) = \frac{1}{\cos x}\,\frac{\cos x \,+\, x\,\sin x}{\cos x} = \frac{\cos x \,+\, x\,\sin x}{\cos^2 x}.$$

The two expressions are equal.

6. State the equation(s) of the vertical asymptote(s) to the curve $y = \dfrac{e^x\,\sin^2 x}{x(x^2-4)}$.

Solution The denominator is 0 when $x = 0$ or $x = \pm 2$. These 3 values can be abscissas of vertical asymptotes. However, they are not automatically. We have to verify whether the limits are infinite.

1) $x = 0$: Separating special limit of $(\sin x)/x$, we obtain

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x}\cdot\frac{e^x\,\sin x}{x^2-4} = 1\cdot\frac{e^0\,\sin 0}{-4} = 1\cdot 0 = 0.$$

Hence there is no asymptote at $x = 0$.

2) $x = 2$: The numerator is $e^2\cdot\sin^2 2 \neq 0$, while the denominator is 0. Therefore, the asymptote $x = 2$ does exist.

3) $x = -2$: Similarly, the numerator is $e^{-2}\cdot\sin^2(-2) \neq 0$, while the denominator is 0. The asymptote $x = 2$ exists.

7. Find equation of the tangent line to the curve $y = 2x^2 - 3x + 2$ at the point where $x = -1$.

Solution At $x = -1$, the function takes value $y = 2\cdot(-1)^2 - 3(-1) + 2 = 7$. The slope of the tangent line is determined from the derivative $y' = 4x - 3$, at $x = -1$ $y' = 4\cdot(-1) - 3 = -7$. Equation of the tangent line is

$$\frac{y-7}{x-(-1)} = -7 \qquad\Longrightarrow\qquad y - 7 = -7(x+1) \qquad\Longrightarrow\qquad y = -7x.$$