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# Otakar Borůvka on minimum spanning tree problem Translation of both the 1926 papers, comments, history

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**Abstract**

Borůvka presented in 1926 the first solution of the Minimum Spanning Tree Problem (MST) which is generally regarded as a cornerstone of Combinatorial Optimization. In this paper we present the first English translation of both of his pioneering works. This is followed by the survey of development related to the MST problem and by remarks and historical perspective. Out of many available algorithms to solve MST the Borůvka's algorithm is the basis of the fastest known algorithms. © 2001 Elsevier Science B.V. All rights reserved.

## 1. Introduction

In the contemporary terminology the Minimum Spanning Tree Problem (shortly, MST Problem) is the following problem:

Given a finite set $V$ and a real weight function $w$ on pairs of elements of $V$, find a tree $(V,T)$ of minimal weight $w(T)=\sum(w(x,y)\colon \{x,y\}\in T)$.

For example when $V$ is a subset of a metric space and the weight function is defined as the distance then a solution $T$ presents the shortest network connecting all points of $V$.

Another formulation, which also explains its name, is the following:

**MST Problem**: Given a connected (undirected) graph $G=(V,E)$ with real weights assigned to its edges. Find a spanning tree $(V,T)$ of $G$ (i.e. $T\subseteq E$) with the minimal weight $w(T)$.

This problem can be found implicitly in various contexts early in the 20th century (see the paper by Graham and Hell [23] for the early history of the problem, see also a follow up by one of the authors in [42]). However, the problem has been solved only in 1926 by Borůvka [1,2]. His formulation given in [2] is as clear as any of the above contemporary formulations:

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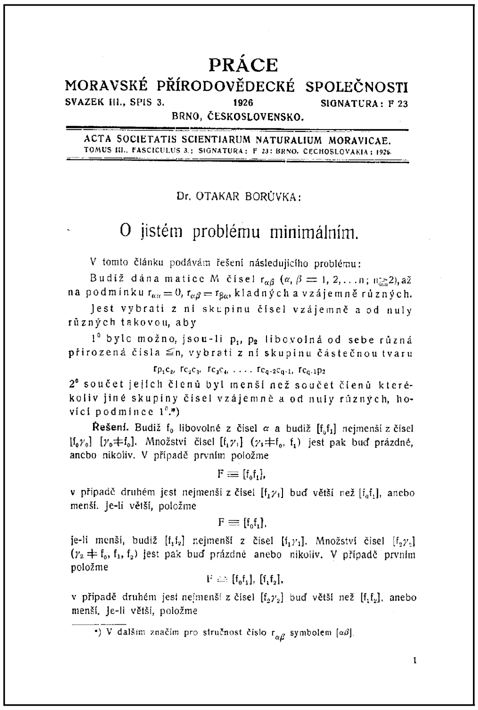
*There are n points given in the plane* (*in the space*) *whose mutual distances are different. The problem is to join them through the net in such a way that*

1. *any two points are joined to each other either directly or by means of some other points*,
2. *the total length of the net would be the smallest.*

The MST problem is a cornerstone of Combinatorial Optimization and in a sense its cradle. The problem is important both in its practical and theoretical applications. Moreover, the recent development places Borůvka's pioneering work in a new and very contemporary context. One can even say that out of many available MST-algorithms Borůvka's algorithm is presently the basis of the fastest known algorithms.

This paper presents the first English translation of both Borůvka's papers. (The original papers are written in Czech, the paper [1] has a six page German summary, the paper [2] is entirely in Czech). We tried to preserve as much of the style of the original articles as possible. This we did not do just for the purpose of the historical accuracy. It is perhaps interesting to compare and to think about the origins and about the style of early years. Rarely we have such a clear and compact possibility.

We aimed for a typotranslation (in the sense of e.g. [16]). Moreover, we included copies of two pages from [1] to give the reader better idea about the original.



PRÁCE
MORAVSKÉ PŘÍRODOVĚDECKÉ SPOLEČNOSTI
SVAZEK III., SPIS 3. 1926 SIGNATURA: F 23
BRNO, ČESKOSLOVENSKO.

ACTA SOCIETATIS SCIENTIARUM NATURALIUM MORAVICAE.
TOMUS III., FASCICULUS 3.; SIGNATURA: F 23; BRNO, ČECHOSLOVAKIA; 1926.

Dr. OTAKAR BORŮVKA:

O jistém problému minimálním.

V tomto článku podávám řešení následujícího problému:

Budiž dána matice M čísel $r_{\alpha\beta}$ ($\alpha, \beta = 1, 2, \ldots n$; $n \geqq 2$), až na podmínku $r_{\alpha\alpha} = 0$, $r_{\alpha\beta} = r_{\beta\alpha}$, kladných a vzájemně různých.

Jest vybrati z ní skupinu čísel vzájemně a od nuly různých takovou, aby

1° bylo možno, jsou-li $p_1$, $p_2$ libovolná od sebe různá přirozená čísla $\leqq n$, vybrati z ní skupinu částečnou tvaru

$$r_{p_1c_2},\ r_{c_2c_3},\ r_{c_3c_4},\ \ldots\ r_{c_{q-2}c_{q-1}},\ r_{c_{q-1}p_2}$$

2° součet jejích členů byl menší než součet členů kterékoliv jiné skupiny čísel vzájemně a od nuly různých, hovící podmínce 1°.*)

**Řešení.** Budiž $f_0$ libovolné z čísel $\alpha$ a budiž $[f_0f_1]$ nejmenší z čísel $[f_0\gamma_0]$ $[\gamma_0 \neq f_0]$. Množství čísel $[f_1\gamma_1]$ $(\gamma_1 \neq f_0, f_1)$ jest pak buď prázdné, anebo nikoliv. V případě prvním položme

$$F \equiv [f_0f_1],$$

v případě druhém jest nejmenší z čísel $[f_1\gamma_1]$ buď větší než $[f_0f_1]$, anebo menší. Je-li větší, položme

$$F \equiv [f_0f_1],$$

je-li menší, budiž $[f_1f_2]$ nejmenší z čísel $[f_1\gamma_1]$. Množství čísel $[f_2\gamma_2]$ $(\gamma_2 \neq f_0, f_1, f_2)$ jest pak buď prázdné anebo nikoliv. V případě prvním položme

$$F \equiv [f_0f_1],\ [f_1f_2],$$

v případě druhém jest nejmenší z čísel $[f_2\gamma_2]$ buď větší než $[f_1f_2]$, anebo menší. Je-li větší, položme

*) V dalším značím pro stručnost číslo $r_{\alpha\beta}$ symbolem $[\alpha\beta]$.

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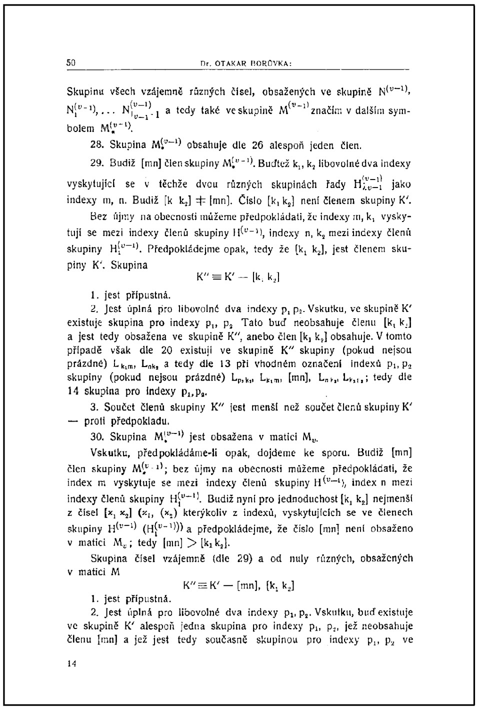

50 Dr. OTAKAR BORŮVKA:

Skupinu všech vzájemně různých čísel, obsažených ve skupině $N^{(v-1)}$, $N_1^{(v-1)}, \ldots N_{l_{v-1}-1}^{(v-1)}$ a tedy také ve skupině $M^{(v-1)}$ značím v dalším symbolem $M_*^{(v-1)}$.

28. Skupina $M_*^{(v-1)}$ obsahuje dle 26 alespoň jeden člen.

29. Budiž [mn] člen skupiny $M_*^{(v-1)}$. Buďtež $k_1$, $k_2$ libovolné dva indexy vyskytující se v těchže dvou různých skupinách řady $H_{\lambda, v-1}^{(v-1)}$ jako indexy m, n. Budiž $[k_1 k_2] \neq [mn]$. Číslo $[k_1 k_2]$ není členem skupiny K′.

Bez újmy na obecnosti můžeme předpokládati, že indexy m, $k_1$ vyskytují se mezi indexy členů skupiny $H^{(v-1)}$, indexy n, $k_2$ mezi indexy členů skupiny $H_1^{(v-1)}$. Předpokládejme opak, tedy že $[k_1 k_2]$, jest členem skupiny K′. Skupina

$$K'' \equiv K' - [k_1 k_2]$$

1. jest přípustná.

2. Jest úplná pro libovolné dva indexy $p_1 p_2$. Vskutku, ve skupině K′ existuje skupina pro indexy $p_1$, $p_2$. Tato buď neobsahuje členu $[k_1 k_2]$ a jest tedy obsažena ve skupině K″, anebo člen $[k_1 k_2]$ obsahuje. V tomto případě však dle 20 existují ve skupině K″ skupiny (pokud nejsou prázdné) $L_{k_1m}$, $L_{nk_2}$ a tedy dle 13 při vhodném označení indexů $p_1$, $p_2$ skupiny (pokud nejsou prázdné) $L_{p_1k_1}$, $L_{k_1m}$, [mn], $L_{nk_2}$, $L_{k_2p_2}$; tedy dle 14 skupina pro indexy $p_1, p_2$.

3. Součet členů skupiny K″ jest menší než součet členů skupiny K′ — proti předpokladu.

30. Skupina $M_*^{(v-1)}$ jest obsažena v matici $M_v$.

Vskutku, předpokládáme-li opak, dojdeme ke sporu. Budiž [mn] člen skupiny $M_*^{(v-1)}$; bez újmy na obecnosti můžeme předpokládati, že index m vyskytuje se mezi indexy členů skupiny $H^{(v-1)}$, index n mezi indexy členů skupiny $H_1^{(v-1)}$. Budiž nyní pro jednoduchost $[k_1 k_2]$ nejmenší z čísel $[\varkappa_1 \varkappa_2]$ ($\varkappa_1$, ($\varkappa_2$) kterýkoliv z indexů, vyskytujících se ve členech skupiny $H^{(v-1)}$ ($H_1^{(v-1)}$)) a předpokládejme, že číslo [mn] není obsaženo v matici $M_v$; tedy $[mn] > [k_1 k_2]$.

Skupina čísel vzájemně (dle 29) a od nuly různých, obsažených v matici M

$$K'' \equiv K' - [mn],\ [k_1 k_2]$$

1. jest přípustná.

2. Jest úplná pro libovolné dva indexy $p_1, p_2$. Vskutku, buď existuje ve skupině K′ alespoň jedna skupina pro indexy $p_1$, $p_2$, jež neobsahuje členu [mn] a jež jest tedy současně skupinou pro indexy $p_1$, $p_2$ ve

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This paper is organized as follows:

1. Introduction.
2. O. Borůvka: On a minimal problem (a typotranslation),
Práce mor. přírodověd. spol. v Brně III, 3 (1926) 37–58 [1].
3. O. Borůvka: Contribution to the solution of a problem of economic construction of electricity power networks (a typotranslation), Elektrotechnický obzor 15 (1926) 153–4 [2].
4. Remarks on O. Borůvka: On a minimal problem [1].
5. Remarks on O. Borůvka: Contribution to the solution of a problem of economic construction of electricity power networks [2].
6. Modern version of Borůvka's article [1].
7. Contemporary formulation of Borůvka's algorithms.
8. History, remarks and perspectives.
9. Appendix: O. Borůvka — life and work.

In Sections 3 and 4 we give some remarks which aid in understanding of historical (pre-algorithmic, pre-graph theory age) Borůvka's text and explain some particular features. Let us just say at this place that Borůvka's rigorous 'mathematical' paper [1] is at some place lengthy and cumbersome and as a result it was nicknamed as 'unnecessarily complicated' (also in view of the particularly elegant later algorithms). However, in preparing his paper [1] Borůvka was honoring the style of his time. He was at the very beginning of his mathematical career and this may help to explain his rather pedantic style (as he communicated to one of the authors [51]). However, he was both convinced about the importance of the work and about the essence of the algorithm. This is documented by his memoirs [3] and, perhaps more importantly, by the fact that he published simultaneously with [1] a short note [2] which is translated in Section 3. This note is little known (e.g. the list of his collected scientific works [39] does not refer to it). In this note, written for the Elektrotechnický obzor (Electrotechnical News) he published a lucid description of his algorithms by means of a geometric example with 40 cities.

In the final two sections first we give a formulation of Borůvka's papers in contemporary language and then trace the influence of his article and MST problem through the history. Particularly, we outline the reasons for recent revival of interest in Borůvka's algorithm.

We end the paper with a brief description of Borůvka's life and works. (Just briefly: he was not 'a Czech engineer' but rather an important and influential mathematician. He died in 1995 at the age of 96.)

## 2. Typotranslation of 'O jistém problému minimálním'

We preserve fully the rather old-fashioned style of this paper. The reader should consult the remarks in Section 4 for explanations and comments and then he/she should compare it with the modern version included in Section 6.

The numbers in brackets $<\mathbf{1}>$, $<\mathbf{2}>$, etc. which are positioned at the beginnings of lines refer to our remarks in Section 4.

## ON A CERTAIN MINIMAL PROBLEM

Otakar Borůvka

In this article I am presenting a solution of the following problem:

*Given a matrix M of numbers* $r_{\alpha\beta}$ $(\alpha,\beta=1,2,\ldots,n;\ n\geqslant 2)$, all positive and pairwise different, with the exception of $r_{\alpha\alpha}=0$ and $r_{\alpha\beta}=r_{\beta\alpha}$.

$<\mathbf{1}>$

*From that matrix a set of nonzero and pairwise different numbers should be chosen such that*

(1) *For any* $p_1$, $p_2$, *mutually different natural numbers* $\leqslant n$, *it would be possible to choose a subset of the form*

$$r_{p_1c_2},\ r_{c_2c_3},\ r_{c_3c_4},\ldots,r_{c_{q-2}c_{q-1}},\ r_{c_{q-1}p_2}.$$

(2) *The sum of its elements would be smaller than the sum of elements of any other subset of nonzero and pairwise different numbers*, *satisfying the condition* (1).[1]

$<\mathbf{2}>$

**Solution**. Let $f_0$ be an arbitrary of the numbers $\alpha$ and let $[f_0f_1]$ be the smallest of the numbers $[f_0\gamma_0]$ $[\gamma_0\neq f_0]$. The set of numbers $[f_1\gamma_1]$ $(\gamma_1\neq f_0,f_1)$ is then either empty or not. In the first case, let us put

$$F\equiv[f_0f_1],$$

in the second case, the smallest of the numbers $[f_1\gamma_1]$ is either greater than $[f_0f_1]$ or smaller. If it is greater, then let us put

$$F\equiv[f_0f_1],$$

if it is smaller, then let $[f_1f_2]$ be the smallest of the numbers $[f_1\gamma_1]$. The set of numbers $[f_2\gamma_2]$ $(\gamma_2\neq f_0,f_1,f_2)$ is either empty or not. In the first case, let us put

$$F\equiv[f_0f_1],\ [f_1f_2],$$

in the second case, the smallest of the numbers $[f_2\gamma_2]$ is either greater than $[f_1f_2]$ or smaller. If it is greater, then let us put

$$F\equiv[f_0f_1],\ [f_1f_2],$$

if it is smaller, then let $[f_2f_3]$ be the smallest of the numbers $[f_2\gamma_2]$. The set of numbers $[f_3\gamma_3]$ $(\gamma_3\neq f_0,f_1,f_2,f_3)$ is either empty or not. In the first case, let us put

$$F\equiv[f_0f_1],\ [f_1f_2],\ [f_2f_3],$$

[1] For the sake of brevity I shall use the symbol $[\alpha\beta]$ instead $r_{\alpha\beta}$ from now on.

in the second case, the smallest of the numbers $[f_3\gamma_3]$ is either greater than $[f_2f_3]$ or smaller. If it is greater, then let us put

$$F \equiv [f_0f_1],\ [f_1f_2],\ [f_2f_3],$$

if it is smaller we shall continue in the same way. Finally, we get a set of numbers

$$F \equiv [f_0f_1],\ [f_1f_2],\ [f_2f_3],\dots,[f_{g-1}f_g].$$

Each of the numbers $1,2,\dots,n$ either occurs among the indices $f_0,f_1,\dots,f_g$ or not.
< **3** >

In the first case, let us put

$$\mathfrak{F} \equiv F,$$

in the second case, let $f_0^{(1)}$ be one of the numbers $1,2,\dots,n$, which does not occur among the numbers $f_0,f_1,\dots,f_g$. Let $[f_0^{(1)}f_1^{(1)}]$ be the smallest of the numbers $[f_0^{(1)}\gamma_0^{(1)}]$ $(\gamma_0^{(1)} \neq f_0^{(1)})$. Considering this number we construct as before a set

$$F_1 \equiv [f_0^{(1)}f_1^{(1)}],[f_1^{(1)}f_2^{(1)}],\dots,[f_{g_1-1}^{(1)}f_{g_1}^{(1)}].$$

During the construction of this set we can come across an element with an index, which occurs among elements of the set $F$; in this case if $f_{h_1}^{(1)}$ is the first index among these indices, we put $f_{g_1}^{(1)} \equiv f_{h_1}^{(1)}$.
< **4** >

Each of the numbers $1,2,\dots,n$ either occurs among the indices $f_0,f_1,\dots,f_g$; $f_0^{(1)},f_1^{(1)},\dots,f_{g_1}^{(1)}$ or not. In the first case, let us put

$$\mathfrak{F} \equiv F,F_1,$$

in the second case, let $f_0^{(2)}$ be one of the numbers $1,2,\dots,n$, which does not occur among the numbers $f_0,f_1,\dots,f_g$; $f_0^{(1)},f_1^{(1)},\dots,f_{g_1}^{(1)}$. Let $[f_0^{(2)}f_1^{(2)}]$ be the smallest of the numbers $[f_0^{(2)}\gamma_0^{(2)}]$ $(\gamma_0^{(2)} \neq f_0^{(2)})$. Considering this number we construct as before a set

$$F_2 \equiv [f_0^{(2)}f_1^{(2)}],[f_1^{(2)}f_2^{(2)}],\dots,[f_{g_2-1}^{(2)}f_{g_2}^{(2)}].$$

During the construction of this set we can come across an element with an index, which occurs among elements of the sets $F,F_1$; in this case if $f_{h_2}^{(2)}$ is the first index among these indices, we put $f_{g_2}^{(2)} \equiv f_{h_2}^{(2)}$.

Each of the numbers $1,2,\dots,n$ either occurs among the indices $f_0,f_1,\dots,f_g$; $f_0^{(1)},f_1^{(1)},\dots,f_{g_1}^{(1)}$; $f_0^{(2)},f_1^{(2)},\dots,f_{g_2}^{(2)}$ or not. In the first case, let us put

$$\mathfrak{F} \equiv F,F_1,F_2,$$

in the second case, we shall continue in the same way. Finally, we get a sequence of sets

$$\mathfrak{F} \equiv F,F_1,F_2,\dots,F_{i-1}.$$

Each of the numbers $1,2,\dots,n$ occurs among the indices of the elements of these sets at least once.
< **5** >

The sequence of sets $\mathfrak{F}$ contains either just the set $F$ or more sets. In the first case, let us put

$$G \equiv F,$$

in the second case, the set $F$ either does not contain an element, the index of which occurs in some of the remaining sets of the sequence $\mathfrak{F}$ or it contains at least one such element. If it does not contain such an element let us put

$$G \equiv F,$$

if it does, let $j$ be the index of a certain element of the set $F$ which occurs at the same time at least in one of the remaining sets of the sequence $\mathfrak{F}$; we may suppose that it occurs at least in the set $F_1$.

The sequence of sets $\mathfrak{F}$ contains either just the sets $F, F_1$ or it contains more sets. In the first case, let us put

$$G \equiv F, F_1,$$

in the second case, the set $F, F_1$ either does not contain an element the index of which occurs in some of the remaining sets of the sequence $\mathfrak{F}$ or it contains at least one such element. If it does not contain such an element let us put

$$G \equiv F, F_1,$$

if it does, let $j_1$ be the index of a certain element of the set $F, F_1$ which occurs at the same time at least in one of the remaining sets of the sequence $\mathfrak{F}$; obviously we may suppose that it occurs at least in the set $F_2$. The sequence of sets $\mathfrak{F}$ contains either just the sets $F, F_1, F_2$ or more sets. In the first case, let us put

$$G \equiv F, F_1, F_2,$$

in the second case, we shall continue in the same way. Finally, we get a set

$$G \equiv F, F_1, F_2, \ldots, F_{k-1}.$$

This set contains either all sets of the sequence $\mathfrak{F}$ or not. In the first case, let us put

$$\mathfrak{G} \equiv G,$$

< **6** >

In the second case, there exists a set $F_k$ of the sequence $\mathfrak{F}$ which does not contain an element the index of which occurs in the set $G$.

The sequence of sets $\mathfrak{F}$ contains either just the sets $G, F_k$ or more sets. In the first case, let us put

$$G_1 \equiv F_k,$$

in the second case, the set $F_k$ either does not contain an element, the index of which occurs in some of the remaining sets of the sequence $\mathfrak{F}$ or it contains at least one such element. If it does not contain such an element let us put

$$G_1 \equiv F_k,$$

if it does, let $j^{(1)}$ be the index of a certain element of the set $F_k$ which occurs at the same time at least in one of the remaining sets of the sequence $\mathfrak{F}$; we may suppose that it occurs at least in the set $F_{k+1}$.

The sequence of sets $\mathfrak{F}$ contains either just the sets $G, F_k, F_{k+1}$ or more sets. In the first case, let us put

$$G_1 \equiv F_k, F_{k+1},$$

in the second case, we shall continue in the same way. Finally, we get a set

$$G_1 \equiv F_k, F_{k+1}, \ldots, F_{k_1-1}.$$

< **7** >

This sequence of sets $G, G_1$ contains either all sets of the sequence $\mathfrak{F}$ or not. In the first case, let us put

$$\mathfrak{G} \equiv G, G_1,$$

in the second case, we shall continue in the same way. Finally, we get a sequence of sets

$$\mathfrak{G} \equiv G, G_1, \ldots, G_{l-1}.$$

The sequence $\mathfrak{G}$ contains all sets of the sequence $\mathfrak{F}$ and no set of the sequence $\mathfrak{G}$ contains an element index of which occurs with an element of another set of this sequence.

< **8** >

Let us put $H_\lambda \equiv G_\lambda$ $(\lambda = 0, 1, \ldots, l-1)$.

The sequence of sets $\mathfrak{G}$ contains either just the set $G$ or more sets. In the first case, let us put

$$J \equiv \mathfrak{G},$$

< **9** >

in the second case, let

$\kappa_\lambda$ be any of the indices which occur in the elements of the set $H_\lambda$;
$\alpha_1, \beta_1$ be two of the numbers $\lambda$;
$[k_{\alpha_1\beta_1} k_{\beta_1\alpha_1}]$ be the smallest of the numbers $[\kappa_{\alpha_1} \kappa_{\beta_1}]$ when $\alpha_1 \neq \beta_1$, $[k_{\alpha_1\beta_1} k_{\beta_1\alpha_1}] = 0$ when $\alpha_1 = \beta_1$;
$M_1$ be the matrix of numbers $[k_{\alpha_1\beta_1} k_{\beta_1\alpha_1}]$ $(\alpha_1, \beta_1 = 0, 1, 2, \ldots, l-1)$;[2]

< **10** >

$\mathfrak{G}_1 \equiv G^{(1)}, G_1^{(1)}, \ldots, G_{l_1-1}^{(1)}$ be the sequence of sets which we get from the matrix $M_1$ in the same way as we got the sequence of sets $\mathfrak{G}$ from the matrix $M$.
$\mathfrak{H}_{\lambda_1}^{(1)}$ be a sequence of those and only those sets chosen from the sequence $H, H_1, \ldots, H_{(l-1)}$, which contains at least one element with the index which occurs at the same time in the set $G_{\lambda_1}^{(1)}$ $(\lambda_1 = 0, 1, \ldots, l_1 - 1)$; put $H_{\lambda_1}^{(1)} \equiv \mathfrak{H}_{\lambda_1}^{(1)}$, $G_{\lambda_1}^{(1)}$.

[2] The matrix $M_1$ is obviously symmetrical, it does not contain any number from the set $\mathfrak{G}$ and its order equals at most to the largest integer $\leqslant n/2$.

Then the sequence $\mathfrak{G}_1$ contains either only one set $G^{(1)}$ or more sets. In the first case, let us put

$$J \equiv \mathfrak{G}, \mathfrak{G}_1,$$

in the second case, let

$\kappa_{\lambda_1}$ be any of the indices which occur in the elements of the set $H_{\lambda_1}^{(1)}$;
$\alpha_2, \beta_2$ be two of the numbers $\lambda_1$;
$[k_{\alpha_2\beta_2} k_{\beta_2\alpha_2}]$ be the smallest of the numbers $[\kappa_{\alpha_2}\kappa_{\beta_2}]$ when $\alpha_2 \neq \beta_2$, $[k_{\alpha_2\beta_2} k_{\beta_2\alpha_2}] = 0$ when $\alpha_2 = \beta_2$;
$M_2$ be the matrix of numbers $[k_{\alpha_2\beta_2} k_{\beta_2\alpha_2}]$ $(\alpha_2, \beta_2 = 0, 1, 2, \ldots, l_1 - 1)$;
$\mathfrak{G}_2 \equiv G^{(2)}$, $G_1^{(2)}, \ldots, G_{l_2-1}^{(2)}$ be the sequence of sets which we get from the matrix $M_2$ in the same way as we got the sequence of sets $\mathfrak{G}$ from the matrix $M$.
$\mathfrak{H}_{\lambda_2}^{(2)}$ be a sequence of those and only those sets chosen from the sequence $H^{(1)}$, $H_1^{(1)}, \ldots, H_{l_1-1}^{(1)}$, which contains at least one element with the index occurring at the same time in the set $G_{\lambda_2}^{(2)}$ $(\lambda_2 = 0, 1, \ldots, l_2 - 1)$;
put $H_{\lambda_2}^{(2)} \equiv \mathfrak{H}_{\lambda_2}^{(2)}, G_{\lambda_2}^{(2)}$.

Then the sequence $\mathfrak{G}_2$ contains either only one set $G^{(2)}$ or more sets. In the first case, let us put

$$J \equiv \mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2,$$

in the second case, let

$\kappa_{\lambda_2}$ be any of the indices which occur in the elements of the set $H_{\lambda_2}^{(2)}$;
$\alpha_3, \beta_3$ be two of the numbers $\lambda_2$;
$[k_{\alpha_3\beta_3} k_{\beta_3\alpha_3}]$ be the smallest of the numbers $[\kappa_{\alpha_3}\kappa_{\beta_3}]$ when $\alpha_3 \neq \beta_3$, $[k_{\alpha_3\beta_3} k_{\beta_3\alpha_3}] = 0$ when $\alpha_3 = \beta_3$;
$M_3$ be the matrix of numbers $[k_{\alpha_3\beta_3} k_{\beta_3\alpha_3}]$ $(\alpha_3, \beta_3 = 0, 1, 2, \ldots, l_2 - 1)$;
$\mathfrak{G}_3 \equiv G^{(3)}, G_1^{(3)}, \ldots, G_{l_3-1}^{(3)}$ be the sequence of sets which we get from the matrix $M_3$ in the same way as we got the sequence of sets $\mathfrak{G}$ from the matrix $M$.
$\mathfrak{H}_{\lambda_3}^{(3)}$ be a sequence of those and only those sets chosen from the sequence $H^{(2)}, H_1^{(2)}, \ldots, H_{l_2-1}^{(2)}$, which contains at least one element with the index occurring at the same time in the set $G_{\lambda_3}^{(3)}$ $(\lambda_3 = 0, 1, \ldots, l_3 - 1)$;
put $H_{\lambda_3}^{(3)} \equiv \mathfrak{H}_{\lambda_3}^{(3)},\ G_{\lambda_3}^{(3)}$.

Then the sequence $\mathfrak{G}_3$ contains either only one set $G^{(3)}$ or more sets. In the first case, let us put

$$J \equiv \mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3,$$

in the second case, we shall continue in the same way. Finally, we get a set

$$J \equiv \mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \ldots, \mathfrak{G}_{u-1},$$

which is a solution of the given problem.
**< 11 >**

**Proof**. To prove this result it suffices to prove the following theorems:

I. *For arbitrary choice of the initial indices of the sets* $F_\alpha$ *of the sequence* $\mathfrak{F}$, *the number* $[mn]$ *from the matrix* $M$ *occurs in some set of this sequence if and only if it is the smallest of either of the numbers* $[m\mu](\mu \neq m)$ *or of the numbers* $[n\nu](\nu \neq n)$.

II. *In the matrix* $M$ *there exists at least one set of nonzero and pairwise different numbers, fulfilling condition* (1) *and such that the sum of its elements is not bigger than the sum of elements of any other group of nonzero and pairwise different numbers, fulfilling condition* (1).

III. *If* $K'$ *is one of the sets with these properties, it contains the sequence of sets* $\mathfrak{G}$.

IV. *If* $u \geqslant 2$ *and* $v \leqslant (u-1)$ *and if the set* $K'$ *contains the sets* $\mathfrak{G}, \mathfrak{G}_1, \ldots, \mathfrak{G}_{v-1}$, *than* $K'$ *contains the set* $\mathfrak{G}_v$.

V. *The set* $K'$ *does not contain an element which is not contained in the set* $J$.

Indeed, then according to I, the set $J$ is fully determined by the matrix $M$ and according to III, IV and V it is identical with every set which has properties of the set $K'$; therefore the set $J$ is the solution of the given problem.

★

★ ★

1. It follows from the construction that the numbers contained in the set $J$ are nonzero and pairwise different; their number is $n-1$.

2. Let $L$ be a set of nonzero and pairwise different numbers contained in the matrix $M$. If and only if the set $L$ contains at least one number from each row of $M$, then I say that $L$ is *admissible*.

3. It follows from the construction that the set $J$ is admissible.

4. For every choice of the initial indices of the sets $F_\alpha$ of the sequence $\mathfrak{F}$ the construction of the set $J$ determines a certain order of elements in each set of this sequence. I call a set $F_p$ of the sequence $\mathfrak{F}$ *ordered* if and only if its elements have this order.

5. For every choice of the initial indices of the sets $F_\alpha$ of the sequence $\mathfrak{F}$ the construction of the set $J$ determines a certain order of sets $F_\alpha$ in the sequence $\mathfrak{F}$. I call the sequence $\mathfrak{F}$ *ordered* if and only if sets $F_\alpha$ have this order.

6. Let

$$F_p \equiv [f_0 f_1], \ldots, [f_k f_{k+1}], \ldots$$

be an ordered set of the sequence $\mathfrak{F}$ with $g(\geqslant 2)$ elements. For a fixed $k$ $(1 \leqslant k \leqslant g-1)$ and $j$ $(1 \leqslant j \leqslant k+1)$ it follows from the construction that

$$[f_j f_\lambda] \geqslant [f_\lambda f_{\lambda+1}] \geqslant [f_k f_{k+1}] \quad (\lambda = 0, \ldots, j-1).$$

**Theorem I**. *For arbitrary choice of the initial indices of sets $F_\alpha$ of the sequence $\mathfrak{F}$, the number $[mn]$ from the matrix $M$ occurs in some set of this sequence if and only if it is the smallest either of the numbers $[m\mu](\mu \neq m)$ or of the numbers $[nv](v \neq n)$.*

1. Let $[mn]$ be an element of the set $F_p$. It suffices to consider the case when the ordered set $F_p$ has the form

$$[f_0 f_1], \dots, [mn], \dots \,.$$

It follows from the construction that $[mn]$ is the smallest of the numbers $[m\gamma](\gamma \neq f_0, \dots, m)$; so according to 6 it is also the smallest of the numbers $[m\mu](\mu \neq m)$.

2. Let $[mn]$ be the smallest of the numbers $[m\mu](\mu \neq m)$. Let $F_p$ be the first set in the ordered sequence of sets $\mathfrak{F}$ which contains the element $[mp]$ with the index $m$. It suffices to consider the case when the set $F_p$ contains at least two elements. There are two and only two mutually exclusive cases:

$m$ is not the last index in the ordered set $F_p$,

$m$ is the last index in the ordered set $F_p$.

In the *first* case, the ordered set $F_p$ has the form

$$[f_0 f_1], \dots, [mp], \dots;$$

thus according to the just derived result $[mp]$ is the smallest of the numbers $[m\mu](\mu \neq m)$ and thus it is identical with $[mn]$.

In the *second* case, the ordered set $F_p$ has the form

$$[f_0 f_1], \dots, [pm]$$

and the set of numbers $[m\gamma](\gamma \neq f_0, \dots, m)$ is either empty or not. If it is empty, it follows from 6 that the number $[pm]$ is smaller than each of the numbers $[mf_\lambda]$ $(f_\lambda \neq p, m)$; thus it is the smallest of the numbers $[m\mu]$ and thus it is identical with the number $[mn]$. If it is not empty, then it follows from the construction that the number $[mp]$ is smaller than the smallest of the numbers $[m\gamma]$ and according to 6 it is also smaller than the smallest of the numbers $[mf_\lambda]$; thus it is the smallest of the numbers $[m\mu]$ and thus it is identical with the number $[mn]$.

$<$ **13** $>$

7. The set $J$ is uniquely determined by the matrix $M$.

This result follows immediately from the construction of the set $J$ and from Theorem I.

$<$ **14** $>$

**Theorem II**. *In the matrix $M$ there exists at least one set of nonzero and pairwise different numbers fulfilling condition* (1) *and such that the sum of its elements is not greater than the sum of elements of any other set of nonzero and pairwise different numbers, fulfilling condition* (1).

Indeed, on the one hand, there is at least one set of nonzero and pairwise different numbers fulfilling condition (1) in the matrix $M$, on the other hand the number of these sets is finite.[3]

< **15** >

8. From now on I shall use the symbol $K'$ for one of the sets with the properties given by Theorem II.

< **16** >

9. Let $L$ be a set of nonzero and pairwise different numbers of the matrix $M$. Let $p_1$ and $p_2$ be two different indices of the elements of the set $L$. I say that the set $L$ is *complete for the indices* $p_1$ and $p_2$ if and only if there is at least one nonempty subset of the set $L$ of the form

$$[p_1q_2], [q_2q_3], \dots, [q_{k-1}p_2].$$

< **17** >

10. The set $L$ is complete for the indices $p_2$ and $p_1$ if and only if it is complete for the indices $p_1$ and $p_2$.

11. The set $L$ fulfills condition (1) if and only if it is admissible and complete for any two indices.

< **18** >

12. Let $L$ be a set of nonzero and pairwise different numbers of the matrix $M$, complete for two indices $p_1$ and $p_2$. Each nonempty subset of the set $L$, which has at least one element of the form

$$[p_1q_2], [q_2q_3], \dots, [q_{k-1}p_2].$$

I shall call *the group for the indices* $p_1, p_2$. If its elements are written exactly in this order, then I shall call it *the ordered group for the indices* $p_1, p_2$.

< **19** >

13. The ordered set for the indices $p_1, p_2$ will be denoted by the symbol $L_{p_1p_2}$. $L_{pq}$ is an empty set if and only if $p=q$. So if $[mn]$ is an element of the set for the indices $p_1, p_2$, then either

$$L_{p_1p_2} \equiv L_{p_1m}, [mn], L_{np_2} \quad \text{or} \quad L_{p_1p_2} \equiv L_{p_1n}, [nm], L_{mp_2}$$

and the sets $L_{p_1m}$ and $L_{np_2}$ or $L_{p_1n}$ and $L_{mp_2}$ do not contain the element $[mn]$.

< **20** >

14. Let $L$ be a set of nonzero and pairwise different numbers of the matrix $M$, complete for any two indices $\varrho_v, \varrho_{v+1}$ $(v = 1, 2, \dots, n-1;\ n \geqslant 3)$.

< **21** >

Let $p_1 \equiv \varrho_1$, $p_2 \equiv \varrho_n$ and $p_1 \neq p_2$. Then the set $L$ is complete for the indices $p_1, p_2$.

Indeed, let us consider a set of numbers

$$\mathfrak{L} \equiv L_{p_1\varrho_2},\ L_{\varrho_2\varrho_3}, \dots, L_{\varrho_{n-1}p_2},$$

[3] It is, for example, the set of nonzero numbers in an arbitrary row of the matrix $M$.

exactly in this order. Let $[p_1q_2]$ be the last number of this set which contains index $p_1$. Then either $q_2 \equiv p_2$ or $q_2 \neq p_2$. In the first case, let us put

$$L_{p_1p_2} \equiv [p_1p_2],$$

in the second case, there is at least one element with index $q_2$ in the set $\mathfrak{L}$ which follows the element $[p_1q_2]$. Let $[q_2q_3]$ be the last number of the set $\mathfrak{L}$ which contains index $q_2$. Then either $q_3 \equiv p_2$ or $q_3 \neq p_2$. In the first case, let us put

$$L_{p_1p_2} \equiv [p_1q_2], [q_2p_2],$$

in the second case, there is at least one element with index $q_3$ in the set $\mathfrak{L}$ which follows the element $[q_2q_3]$. Let $[q_3q_4]$ be the last number of the set $\mathfrak{L}$ which contains index $q_3$. Then either $q_4 \equiv p_2$ or $q_4 \neq p_2$. In the first case, let us put

$$L_{p_1p_2} \equiv [p_1q_2], [q_2q_3], [q_3p_2],$$

in the second case, we shall proceed in the same way. After finite number of steps we evidently get an ordered set for indices $p_1, p_2$ contained in the set $L$.

15. Let $L$ be a set of nonzero and pairwise different numbers of the matrix $M$. Let $L^*$ be a subset of the set $L$. I shall use the symbol $L - L^*$ for the set of all numbers contained in the set $L$ but not contained in the set $L^*$.

**< 22 >**

**Theorem III**. *The set $K'$ contains the sequence of sets $\mathfrak{G}$.*

The theorem evidently holds if $M$ is a matrix of order 2 or 3. So let $n \geqslant 4$. To get a contradiction, assume that the theorem is not true. Indeed, let $[mn]$ be a number of the matrix $M$ which occurs in the sequence $\mathfrak{G}$ and does not occur in the set $K'$. It follows from the construction that the set $\mathfrak{G}$ is identical with the set $\mathfrak{F}$. It follows then from Theorem I that $[mn]$ is either the smallest of the numbers $[m\mu](\mu \neq m)$ or the smallest of the numbers $[nv](v \neq n)$. Without loss of generality, we may assume that it is the smallest of the numbers $[m\mu]$. The set $K'$ necessarily contains the element $[mp]$ with the index $m$. According to the assumption $[mp]$ is not identical with the number $[mn]$; so it must be greater.

The set $K'$ is complete for the indices $m, n$; so there are two and only two mutually exclusive cases:

Each subset of the set $K'$ for the indices $m, n$ contains the number $[mp]$.

There is at least one set for the indices $m, n$ in the set $K'$ which does not contain the number $[mp]$.

**< 23 >**

In the *first case*, there is a set for the indices $m, n$ within the set $K'$ which according to 13 can be written in the form

$$[mp], L_{pn}$$

and the set $L_{pn}$ for the indices $p, n$ necessarily contains at least one element. The set $K'' \equiv K' - [mp], [mn]$

**< 24 >**

of nonzero and pairwise different numbers of the matrix $M$

1. is admissible;
2. is complete for any two indices $p_1, p_2$. Indeed, either there exist at least one set for the indices $p_1, p_2$ in the set $K'$ which does not contain the element $[mp]$ and thus it is also the set for the indices $p_1, p_2$ in the set $K''$ or each set for the indices $p_1, p_2$ in the set $K'$ contains the element $[mp]$. But in this case, in a proper notation of both the indices $p_1, p_2$, within the set $K''$ there evidently exist sets (if nonempty) $L_{p_1m}$, $[mn]$, $L_{np}$, $L_{pp_2}$ and thus according to 14 there exists the set for the indices $p_1, p_2$.
3. the sum of elements of the set $K''$ is less than the sum of elements of the set $K'$ — which is a contradiction.

In the *second case*, there exists at least one set for the indices $m, n$ in the set $K'$ which can be written in the form

$$[mq],\ L_{qn}.$$

According to the assumption necessarily $q \neq n$, and thus $[mq] > [mn]$. The set $L_{qn}$ for the indices $q, n$ necessarily contains at least one element. It suffices to apply the above-described reasoning for the set

$$K'' \equiv K' - [mq], [mn].$$

**< 25 >**

16. Let $L$ be a set of nonzero and pairwise different numbers of the matrix $M$. The set $L$ is not complete for every two indices if and only if it is possible to split the set $L$ into two nonempty subsets $L_1, L_2$ whose union is $L$ and such that none of the subsets $L_1, L_2$ contains an element with an index occurring also in the other subset.

**< 26 >**

1. Let $L_1, L_2$ be two sets of the above-described properties, let $[p_1q_2']$ be an element of the set $L_1$, let $[q_{k-1}'p_2]$ be an element of the set $L_2$ and let us assume that in the set $L$ there exists at least one set for the indices $p_1, p_2$

$$L_{p_1p_2} \equiv [p_1q_2], [q_2q_3], \ldots, [q_{k-1}p_2].$$

Because according to the assumption the set $L_2$ contains no element with the index $p_1$, the element $[p_1q_2]$ is necessarily contained in the set $L_1$. In a similar way, we could show that the set $L_1$ also contains all other elements of the set $L_{p_1p_2}$, especially the element $[q_{k-1}p_2]$. Thus, both of the sets $L_1, L_2$ contain an element with the index $p_2$ — which is a contradiction.

2. Let $[p_1q_2], [q_{k-1}p_2]$ be two elements of the set $L, p_1 \neq p_2$ and let us assume that the set $L$ is not complete for the indices $p_1, p_2$. Let us denote $\mathfrak{L}_1 \equiv [p_1q_2]$. The set $L - \mathfrak{L}_1$ either does not contain an element the index of which occurs also in the set $\mathfrak{L}_1$ or it contains at least one such element. In the first case, let

us put

$$L_1 \equiv \mathfrak{L}_1, \qquad L_2 \equiv L - \mathfrak{L}_1,$$

in the second case, let $\mathfrak{L}_2$ be the subset of the set $L - \mathfrak{L}_1$ which contains elements with indices occurring at the same time in the set $\mathfrak{L}_1$. The set $L - \mathfrak{L}_1 - \mathfrak{L}_2$ either does not contain an element with index occurring at the same time in the set $\mathfrak{L}_1$, $\mathfrak{L}_2$ or it contains at least one such element. In the first case, let us put

$$L_1 \equiv \mathfrak{L}_1, \mathfrak{L}_2, \qquad L_2 \equiv L - \mathfrak{L}_1 - \mathfrak{L}_2,$$

in the second case, let $\mathfrak{L}_3$ be the subset of the set $L - \mathfrak{L}_1 - \mathfrak{L}_2$ which contains elements with indices occurring at the same time in the sets $\mathfrak{L}_1$, $\mathfrak{L}_2$. The set $L - \mathfrak{L}_1 - \mathfrak{L}_2 - \mathfrak{L}_3$ either does not contain an element with index occurring at the same time in the sets $\mathfrak{L}_1$, $\mathfrak{L}_2$, $\mathfrak{L}_3$ or it contains at least one such element. In the first case, let us put

$$L_1 \equiv \mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \qquad L_2 \equiv L - \mathfrak{L}_1 - \mathfrak{L}_2 - \mathfrak{L}_3,$$

in the second case, we continue as before. We shall evidently get two sets $L_1$, $L_2$ such that none of them contains an element with index occurring in the other set. The set $L_1$ evidently contains at least one element. The set $L_2$ also contains at least one element. Indeed, it follows from the construction that the set $L_1$ is complete for any two indices; thus it does not contain the element $[q_{k-1}p_2]$. Thus the element $[q_{k-1}p_2]$ is contained in the set $L_2$.

**< 27 >**

17. Let $L$ be a set of nonzero and pairwise different numbers of the matrix $M$. Let $L_1, L_2$ be nonempty subsets of the set $L$ whose union is $L$ and such that none of the subsets $L_1, L_2$ contains an element with an index occurring also in the other subset. Let $L^*$ be a nonempty subset of the set $L$, complete for any two indices. One of the sets $L_1, L_2$ contains the whole set $L^*$.

This theorem follows immediately from 16.

18. Each set $F_\iota$ ($\iota$ fixed, $\leqslant i-1$) of the sequence $\mathfrak{F}$ is complete for any two indices $p_1$, $p_2$.

It evidently suffices to consider the set $F$. Let $p_1 \equiv f_h p_2 \equiv f_k$ $(h,k \leqslant g;\ h \neq k)$.

**< 28 >**

Then either $h < k \leqslant g$ or $k < h \leqslant g$.

In the first case,

$$[p_1 f_{h+1}],\ [f_{h+1} f_{h+2}], \ldots, [f_{k-1} p_2]$$

is the ordered set for the indices $p_1, p_2$, in the second case,

$$[p_1 f_{h-1}],\ [f_{h-1} f_{h-2}], \ldots, [f_{k+1} p_2]$$

is the ordered set for the indices $p_1, p_2$.

19. Each set $G_\lambda$ ($\lambda$ fixed, $\leqslant l-1$) of the sequence $\mathfrak{G}$ is complete for any two indices $p_1$, $p_2$.

It evidently suffices to consider the set $G$. If $G \equiv F$ then the theorem is true according to 18.

So let $G \equiv F,\ F_1,\dots,F_{k-1}$ $(k \geqslant 2)$. Let us put

$F^{(\kappa-1)} \equiv F,\ F_1,\dots,F_{\kappa-1} \quad (\kappa \leqslant k).$

**Proof by induction**. Let us assume that each set of the sets $F,\ F^{(1)},\dots,F^{(m-1)}$ ($m$ fixed, $\leqslant k-1$) is complete for any two indices $p_1, p_2$. We shall show that then also the set $F^{(m)}$ is complete for any two indices.

It suffices to consider the case in which the index $p_1$ occurs only among the indices of the elements of the set $F^{(m-1)}$, the index $p_2$ occurs only among the indices of the elements of the set $F_m$. It follows from the construction of the set $G$ that in the set $F^{(m-1)}$ there is an element with the index $j_{m-1}$ which occurs also among the indices of the elements of the set $F_m$; so it holds that $p_1 \neq j_{m-1}, p_2 \neq j_{m-1}$. According to the assumption the set $F^{(m-1)}$ and thus also the set $F^{(m)}$ are complete for the indices $p_1$, $j_{m-1}$ and according to 18 the set $F_m$ and thus also the set $F^{(m)}$ are complete for the indices $j_{m-1},\ p_2$. Thus according to 14 the set $F^{(m)}$ is complete for the indices $p_1, p_2$.

**< 29 >**

20. Each set $H^{(\varrho)}_{\lambda_\varrho}$ ($\varrho$ fixed, $\leqslant u-1$; $\lambda_\varrho$ fixed, $\leqslant l_\varrho - 1$) is complete for any two indices $p_1, p_2$.

**< 30 >**

It evidently suffices to consider the set $H^{(\varrho)}$. If $\varrho = 0$ $(H^{(0)} \equiv H \equiv G)$ then according to 19 the theorem is true. So let $\varrho \geqslant 1$.

**Proof by induction**. Let us assume that each set of the sequence $H_\lambda, H^{(1)}_{\lambda_1},\ \dots,\ H^{(m-l)}_{\lambda_{m-1}}$ ($m$ fixed, $\leqslant \varrho$; $\lambda = 0,1,\dots,l-1$; $\lambda_1 = 0,1,\dots,l_1-1$; $\lambda_{m-1} = 0,1,\dots,l_{m-1}-1$) is complete for any two indices $p_1, p_2$ and we shall show that then also the set $H^{(m)}$ is complete for any two indices.

If the set $H^{(m)}$ would not be complete for any two indices then according to 16 it would be possible to split it into two nonempty subsets $L_1, L_2$ whose union is $L$ and such that none of the subsets $L_1, L_2$ contains an element with an index occurring also in the other subset. It follows from the construction that $H^{(m)} \equiv \mathfrak{H}^{(m)},\ G^{(m)}$; thus one of the sets $L_1, L_2$ would contain at least one element of the set $\mathfrak{H}^{(m)}$, thus it would contain at least one element of a certain set of the sequence $H^{(m-l)}_{\lambda_{m-1}}$ and thus according to 17 it would contain the whole set; thus according to the construction it would contain at least one element of the set $G^{(m)}$ and thus according to 19 and 17 it would contain the whole set; thus according to the construction it would contain at least one element of each of the remaining sets of the sequence $\mathfrak{H}^{(m)}$ and thus according to 17 it would contain this whole sequence of sets; thus the other of the sets $L_1, L_2$ would be empty — which is a contradiction.

21. The set $J$ fulfills condition (1).

This theorem follows immediately from 3, 20 and 11.

22. It follows from the construction that the sequence of sets $H^{(\varrho)}_{\lambda_\varrho}$ ($\varrho$ is fixed, $\leqslant u-1$; $\lambda_\varrho = 0,1,\dots,l_\varrho - 1$) contains exactly all numbers which are contained in the sequence $\mathfrak{G}, \mathfrak{G}_1,\dots,\mathfrak{G}_\varrho$ and nothing else.

23. For $u \geqslant 2$ it follows from the construction that no set of the sequence of sets $H_{\lambda_\varrho}^{(\varrho)}$ ($\varrho$ is fixed, $\leqslant u-2$; $\lambda_\varrho = 0, 1, \ldots, l_\varrho - 1$) contains an element with an index occurring in another set of the same sequence.

$< \mathbf{31} >$

24. Let $u \geqslant 2$; $v \leqslant u-1$. If the set $K'$ contains sets $\mathfrak{G}, \mathfrak{G}_1, \ldots, \mathfrak{G}_{v-1}$ then according to 22 it contains the sequence of sets $H_{\lambda_{v-1}}^{(v-l)}$; thus it has the form

$$K' \equiv H^{(v-1)},\ H_1^{(v-1)}, \ldots, H_{l_{v-1}-1}^{(v-1)}; M^{(v-1)}.$$

25. It follows immediately from 23 and 16 that the set $M^{(v-1)}$ is not empty.

26. For each set of the sequence $H_{\lambda_{v-1}}^{(v-1)}$ there is at least one element in the set $M^{(v-1)}$ which contains exactly one index occurring among the indices of the elements of this set.

Indeed, otherwise it would follow immediately from 23 and 16 that the set $K'$ is not complete for any two indices.

27. Let $N_{\lambda_{v-1}}^{(v-1)}$ be the set of all those elements of the set $M^{(v-1)}$ whose exactly one index occurs among the indices of the elements of the set $H_{\lambda_{v-1}}^{(v-1)}$ ($\lambda_{v-1}$ fixed).

From now on I use the symbol $M_*^{(v-1)}$ for the set of pairwise different numbers occurring in the set $N^{(v-1)}, N_1^{(v-1)}, \ldots, N_{l_{v-1}-1}^{(v-1)}$ and thus also in the set $M^{(v-1)}$.

28. According to 26 the set $M_*^{(v-1)}$ contains at least one element.

29. Let $[mn]$ be an element of the set $M_*^{(v-1)}$. Let $k_1, k_2$ be arbitrary two indices which occur as indices $m, n$ in the same two different sets of the sequence $H_{\lambda_{v-1}}^{(v-1)}$. Let $[k_1 k_2] \neq [mn]$. The number $[k_1 k_2]$ is not an element of the set $K'$.

$< \mathbf{32} >$

Without loss of generality, we can assume that the indices $m, k_1$ occur among the indices of elements of the set $H^{(v-1)}$, the indices $n, k_2$ occur among the indices of elements of the set $H_1^{(v-1)}$. Let us assume that on the contrary $[k_1 k_2]$ is an element of the set $K'$. The set

$$K'' \equiv K' - [k_1 k_2]$$

1. is admissible;

2. is complete for any two indices $p_1, p_2$. Indeed, there is a set for the indices $p_1, p_2$ in the set $K'$. This set either does not contain the element $[k_1 k_2]$ and thus it is contained in the set $K''$ or it contains the element $[k_1 k_2]$. But in this case according to 20 in the set $K''$ there exist sets (if nonempty) $L_{k_1 m}$, $L_{n k_2}$ and thus according to 13 in a proper notation of the indices $p_1, p_2$ there are sets (if nonempty) $L_{p_1 k_1}, L_{k_1 m}, [mn], L_{n k_2}, L_{k_2 p_2}$; thus according to 14 there exists the set for the indices $p_1, p_2$ in the set $K''$;

3. the sum of elements of the set $K''$ is less than the sum of elements of the set $K'$ — which is a contradiction.

30. The set $M_*^{(v-1)}$ is contained in the matrix $M_v$. Indeed, if we assume the contrary, we will get a contradiction. Let $[mn]$ be an element of the set $M_*^{(v-1)}$; without loss of generality, we can assume that the index $m$ occurs among the indices of elements of the set $H^{(v-1)}$, the index $n$ occurs among the indices of elements of the set $H_1^{(v-1)}$.

For the sake of simplicity, let $[k_1k_2]$ be the smallest of the numbers $[\kappa_1\kappa_2]$ ($\kappa_1$, ($\kappa_2$, respectively) is any of the indices occurring in elements of the set $H^{(v-1)}$ ($H_1^{(v-1)}$, respectively)) and let us assume that the number $[mn]$ is not contained in the matrix $M_v$; thus $[mn] > [k_1k_2]$.

The set $K'' \equiv K' - [mn], [k_1k_2]$ of nonzero and pairwise different numbers of the matrix $M$ (according to 29)

1. is admissible.
2. is complete for any two indices $p_1, p_2$. Indeed, either there is at least one set for indices $p_1, p_2$ in the set $K'$ which does not contain the element $[mn]$ and thus it is also the set for the indices $p_1, p_2$ in the set $K''$ or each set for the indices $p_1, p_2$ in the set $K'$ contains the element $[mn]$. But in this case there exist according to 20 sets (if nonempty) $L_{mk_1}, L_{k_2n}$ in the set $K''$ and thus according to 13 in a suitable notation for the indices $p_1, p_2$ in the set $K''$ there exist sets (if nonempty) $L_{p_1m}, L_{mk_1}, [k_1k_2], L_{k_2n}, L_{np_2}$; thus according to 14 there is the set for the indices $p_1, p_2$ in the set $K''$.
3. The sum of elements of the set $K''$ is less than the sum of elements of the set $K'$ — which is a contradiction.

**Theorem IV**. *Let $u \geqslant 2$, $v \leqslant u-1$. If the set $K'$ contains the sequence $\mathfrak{G}, \mathfrak{G}_1, \ldots, \mathfrak{G}_{v-1}$, than $K'$ contains the set $\mathfrak{G}_v$.*

Indeed, if the set $K'$ contains the sequence $\mathfrak{G}, \mathfrak{G}_1, \ldots, \mathfrak{G}_{v-1}$, then according to 24, 27, 28, 30 it contains the nonempty set of numbers $M_*^{(v-1)}$ which is contained in the matrix $M_v$.

The set $M_*^{(v-1)}$ of nonzero and pairwise different numbers contained in the matrix $M_v$

1. is admissible for the matrix $M_v$ according to 26;
2. is complete for any two indices. Indeed, otherwise it would be possible (according to 16) to split it into two nonempty subsets whose union is the whole set and none of the subsets contains an element with an index occurring also in the other subset. It would follow immediately from 23 and 16 that the set $K'$ is not complete for any two indices — which is a contradiction;
3. obviously, the sum of elements of the set $M_*^{(v-1)}$ is not greater than the sum of elements of any other set contained in the matrix $M_v$ and fulfilling two preceeding conditions.

Thus, the set $M_*^{(v-1)}$ is the set of numbers of the matrix $M_v$ which have the same properties as the set of numbers $K'$ of the matrix $M$. Thus by the Theorem III it contains the set $\mathfrak{G}_v$.

31. The set $K'$ contains the set $J$.

This theorem follows immediately from Theorems III and IV.

**Theorem V**. *The set $K'$ does not contain an element which is not contained in the set $J$.*

Indeed, according to 21, the set $J$ fulfills condition (1). Thus the sum of its elements cannot be less than the sum of elements of the set $K'$.

32. The set $J$ is the solution of the given problem.

This theorem follows immediately from 7, 31 and Theorem V.

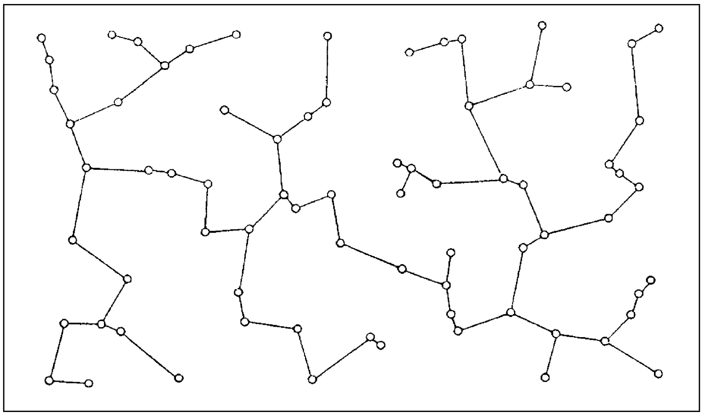
Note: If the numbers $[\alpha\beta]$ of the matrix $M$ fulfill special conditions, we can interpret them as distances among $n$ points; with regard to the solution described above the following problem can be solved:

*Let* $n$ $[\geqslant 2]$ *points be given in the plane* (*generally in the* $r$*-dimensional space*) *whose mutual distances are different. The problem is to join them by a net such that*

(1) *every two points are joined either directly or through some other points*,
(2) *the length of the whole net is minimum.*

**< 33 >**

In the following picture one can see the solution of this problem for a special case.[4]



**< 34 >**

## 3. Typotranslation of 'Příspěvek k řešení otázky ekonomické stavby elektrovodních sítí'

A CONTRIBUTION TO THE SOLUTION OF A PROBLEM OF ECONOMIC CONSTRUCTION OF POWER-NETWORKS

Dr. Otakar Borůvka

In my paper *On a certain minimal problem* I proved a general theorem, which, as a special case, solves the following problem:

[4] It is explained in my paper 'A contribution to the solution of a problem of economic construction of power-network' in Elektrotechnický obzor 15, 1926 how (based on the result of this paper) one can find the solution effectively.

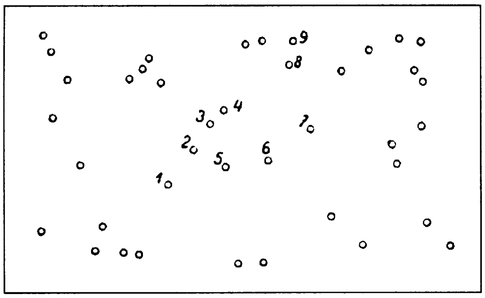



Fig. 1.

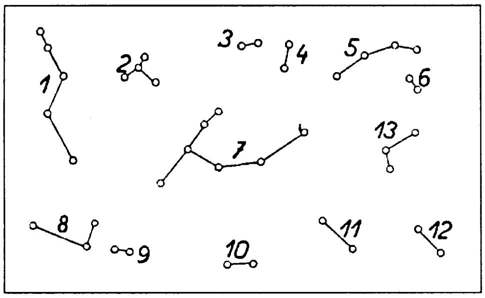



Fig. 2.

*There are n points given in the plane* (*in the space*) *whose mutual distances are all different. We wish to join them by a net such that*

1. *any two points are joined either directly or by means of some other points*,
2. *the total length of the net would be the shortest possible.*

It is evident that a solution of this problem could have some importance in electricity power network designs; hence I present the solution briefly using an example. The reader with a deeper interest in the subject is referred to the above quoted paper.

I shall give the solution of the problem in the case of 40 points given in Fig. 1.

I shall join each of the given points with the nearest neighbour. Thus, for example, point 1 with point 2, point 2 with point 3, point 3 with point 4 (point 4 with point 3), point 5 with point 2, point 6 with point 5, point 7 with point 6, point 8 with point 9 (point 9 with point 8), etc. I shall obtain a sequence of polygonal strokes $1,2,\ldots,13$ (Fig. 2).

I shall join each of these strokes with the nearest stroke in the shortest possible way. Thus, for example, stroke 1 with stroke 2, (stroke 2 with stroke 1) stroke 3 with stroke 4, (stroke 4 with stroke 3), etc. I shall obtain a sequence of polygonal strokes $1,2,\ldots,4$ (Fig. 3). I shall join each of these strokes in the shortest way with the nearest

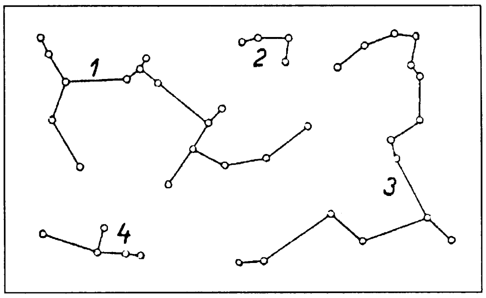



Fig. 3.

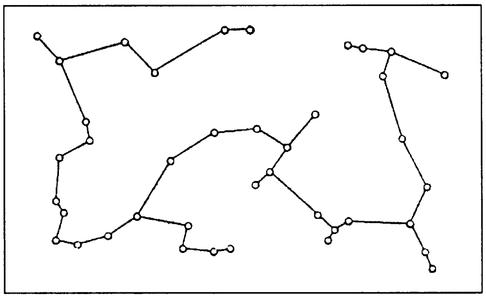


Fig. 4.

stroke. Thus stroke 1 with stroke 3, stroke 2 with stroke 3 (stroke 3 with stroke 1), stroke 4 with stroke 1. I shall finally obtain a single polygonal stroke (Fig. 4), which solves the given problem.

## 4. Remarks on Borůvka 'O jistém problému minimálním' [1]

For the sake of historical accuracy, we did not try to modernize the original text. Instead we tried to keep as close to the original as possible. Here are some of very few linguistic transpositions which were necessary and which we want to mention explicitly:

The Borůvka term 'řádek' or 'řada' (= Czech for 'row', 'sequence') is used thorough out the text. For better understanding we are translating this as

(a) row (in a matrix context),
(b) sequence,
(c) set (when only membership is used).

Another frequently used word is ‘grupa’ (= group). This we translate as set, sequence, collection.

We note that the word set (in the Czech equivalents ‘množina’ or old-fashioned ‘množství’) is never used in the whole of Borůvka’s paper.

Here are some critical and explanatory remarks to the text. These refer to the numbering $<\mathbf{i}>$ in the text.

$<\mathbf{1}>$ In the present interpretation of MST problem we see that the number $r_{\alpha\beta}$ of the matrix $M$ denote the weight of edge $[\alpha,\beta]$ of complete graph $K_n$ (with vertices $1,2,\ldots,n$). Indices $\alpha,\beta$ correspond to the vertices of $K_n$. Thus the word ‘index’ in most cases refers to a vertex of a graph. Borůvka assumes from the very beginning that all the weights are distinct. This assumption is not justified. Borůvka as an analyst was aware of perturbation argument [4] and today this is an assumption which is even easier to satisfy by any tie-breaking procedure (for example we list all weights and in the case that two weights are equal the first weight on our list is bigger).

$<\mathbf{2}>$ A remark is needed at the very beginning: Although Borůvka’s motivation was geometric (as clearly documented by [2]), his paper is written in algebraic language. There are no notions of neighborhood, connectivity, tree, graph. We can also read the paper [1] as a witness (and an apotheosis) of the effectivity of graph-theory language (which was mostly developed after 1926 and which was then not yet related to optimization problems).

$<\mathbf{3}>$ The set $F$ corresponds to a simple path $f_0f_1, f_1f_2,\ldots,f_{g-1}f_g$ where each $f_if_{i+1}$ is the edge of the smallest weight incident with $f_i$.

$<\mathbf{4}>$ Similarly as in the above remark $<\mathbf{3}>$ the set $F_1$ corresponds to a simple path from the vertex $f_o^{(1)}$ to the vertex $f_{g_1}^{(1)}$ (with the same properties as the set $F$). This path $F_1$ can be attached to the path $F$ at the vertex $F_{h_1}^{(1)}$.

$<\mathbf{5}>$ The set $\mathfrak{F}$ corresponds to $i$ simple paths $F, F_1,\ldots,F_{i-1}$. Some of them, or all, can have common vertices. Thus $\mathfrak{F}$ corresponds to a forest. Borůvka now explicitly and elaborately describes the components $G, G_1,\ldots,G_{l-1}$ of this forest.

$<\mathbf{6}>$ If $G$ contains all sets of the sequence $\mathfrak{F}$ then $G$ corresponds to the spanning tree of $K_n$.

$<\mathbf{7}>$ I.e. $G_1$ is the second component of the forest formed by the paths $F, F_1,\ldots,F_{i-1}$.

$<\mathbf{8}>$ This completes the description of the first step in Borůvka’s algorithm: $\mathfrak{G}$ is the forest which we get by joining each vertex to its nearest neighbour.

How difficult it is to formalize this step without using the word ‘tree’!

$<\mathbf{9}>$ Thus $J$ is the desired solution.

$<\mathbf{10}>$ This completes the description of the second step (contraction) in Borůvka’s algorithm.

$<\mathbf{11}>$ $J$ is the (uniquely determined) final tree. This is the end of Borůvka’s algorithm. The brevity of the description of sets $\mathfrak{G}_1, \mathfrak{G}_2,\ldots$ indicates that the author was well aware of contraction- and recursive-part of algorithm.

Now Borůvka proves the correctness of the algorithm.

$<\mathbf{12}>$ What follows are introductory remarks and definitions.

Further remarks, definitions and propositions accompanying the text are numbered by $7, 8, \ldots, 25$. As always we preserve the author's style.

Borůvka proceeds by stating and proving Theorems I–V (stated at the beginning of the proof).

$<$ **13** $>$ This ends the proof of Theorem I. It follows a remark.

$<$ **14** $>$ Proof of Theorem II.

$<$ **15** $>$ End of proof of Theorem II. It follows a chain of Remarks 8–15.

$<$ **16** $>$ I.e. $K'$ is any solution of the MST problem.

$<$ **17** $>$ Today we would simply say that $L$ connects $p_1$ and $p_2$, or that $p_1$ and $p_2$ belong to the same component of $L$.

$<$ **18** $>$ I.e. $L$ is connected and spanning iff (1).

$<$ **19** $>$ The set of indices $p_1 p_2$ is of course a path from $p_1$ to $p_2$.

$<$ **20** $>$ What is meant here is that any path $L_{p_1 p_2}$ containing $[m, n]$ can be written in this way.

$<$ **21** $>$ This should mean: If pairs $p_1 = q_1 q_2, q_2 q_3, \ldots, q_{n-1} p_2$ are in the same component of $L$ then also $p_1$ and $p_2$ are in the same component.

$<$ **22** $>$ End of remarks and definitions. Now the key part of the proof.

$<$ **23** $>$ Recall: complete means connected.

$<$ **24** $>$ Read: $K'' = (K' - [mp]) \cup [mp]$.

As everywhere we preserve all author's types.

$<$ **25** $>$ This ends the Proof of Theorem III.

Twice we have here the exchange axiom in a rudimental form. No bases and circuits are mentioned, yet the key formula is displayed. What follows is a sequence of remarks and definitions.

$<$ **26** $>$ I.e. any not connected graph has at least two components. What follows is proof divided into two steps.

$<$ **27** $>$ End of proof of 16.

$<$ **28** $>$ $F$ is the first group defined at the beginning of the algorithm.

$<$ **29** $>$ So $G_i$ are connected subgraphs of $\mathfrak{G}$.

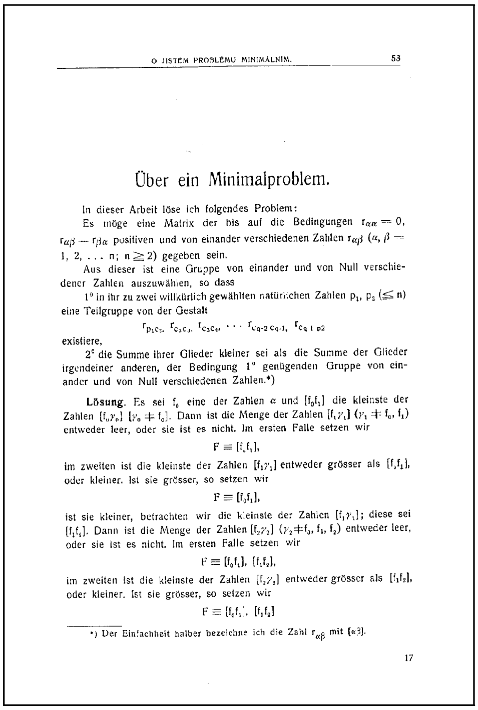
$<$ **30** $>$ So all the sets created in the algorithm are connected (this seems to be the crucial difficulty in Borůvka's writing: he tries to control connectivity at each step).

$<$ **31** $>$ Of course the recursive nature of Borůvka's algorithm could not be well understood (in 1926). What follows is discussion of another minimal spanning tree denoted by $K'$.

$<$ **32** $>$ This is anticipating the statement of Theorem IV.

$<$ **33** $>$ The example given here is the same as the one analyzed in [2]. However, the example given here is in the correct position while in [2] it is reversed.

$<$ **34** $>$ What follows is six pages of German summary. This is the translation of the beginning of the article up to statements I–V (which follow our remark $<$ **11** $>$. We included here copies of the first and the last page of this translation.



O JISTÉM PROBLÉMU MINIMÁLNÍM. 53

Über ein Minimalproblem.

In dieser Arbeit löse ich folgendes Problem:

Es möge eine Matrix der bis auf die Bedingungen $r_{\alpha\alpha} = 0$, $r_{\alpha\beta} = r_{\beta\alpha}$ positiven und von einander verschiedenen Zahlen $r_{\alpha\beta}$ ($\alpha, \beta = 1, 2, \ldots n$; $n \geqq 2$) gegeben sein.

Aus dieser ist eine Gruppe von einander und von Null verschiedener Zahlen auszuwählen, so dass

1° in ihr zu zwei willkürlich gewählten natürlichen Zahlen $p_1, p_2$ ($\leqq n$) eine Teilgruppe von der Gestalt

$$r_{p_1c_2}, r_{c_2c_3}, r_{c_3c_4}, \ldots r_{c_{q-2}c_{q-1}}, r_{c_{q-1}p_2}$$

existiere,

2° die Summe ihrer Glieder kleiner sei als die Summe der Glieder irgendeiner anderen, der Bedingung 1° genügenden Gruppe von einander und von Null verschiedenen Zahlen.*)

**Lösung.** Es sei $f_0$ eine der Zahlen $\alpha$ und $[f_0f_1]$ die kleinste der Zahlen $[f_0\gamma_0]$ [$\gamma_0 \neq f_0$]. Dann ist die Menge der Zahlen $[f_1\gamma_1]$ ($\gamma_1 \neq f_0, f_1$) entweder leer, oder sie ist es nicht. Im ersten Falle setzen wir

$$F \equiv [f_0f_1],$$

im zweiten ist die kleinste der Zahlen $[f_1\gamma_1]$ entweder grösser als $[f_0f_1]$, oder kleiner. Ist sie grösser, so setzen wir

$$F \equiv [f_0f_1],$$

ist sie kleiner, betrachten wir die kleinste der Zahlen $[f_1\gamma_1]$; diese sei $[f_1f_2]$. Dann ist die Menge der Zahlen $[f_2\gamma_2]$ ($\gamma_2 \neq f_0, f_1, f_2$) entweder leer, oder sie ist es nicht. Im ersten Falle setzen wir

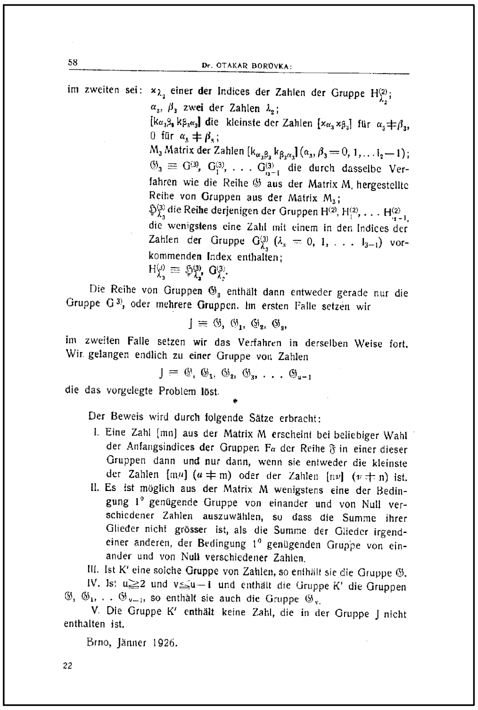
$$F \equiv [f_0f_1], [f_1f_2],$$

im zweiten ist die kleinste der Zahlen $[f_2\gamma_2]$ entweder grösser als $[f_1f_2]$, oder kleiner. Ist sie grösser, so setzen wir

$$F \equiv [f_0f_1], [f_1f_2]$$

*) Der Einfachheit halber bezeichne ich die Zahl $r_{\alpha\beta}$ mit $[\alpha\beta]$.

17



58 Dr. OTAKAR BORŮVKA:

im zweiten sei: $\varkappa_{\lambda_2}$ einer der Indices der Zahlen der Gruppe $H^{(2)}_{\lambda_2}$;
$\alpha_3, \beta_3$ zwei der Zahlen $\lambda_2$;
$[k_{\alpha_3\beta_3} k_{\beta_3\alpha_3}]$ die kleinste der Zahlen $[\varkappa_{\alpha_3}\varkappa_{\beta_3}]$ für $\alpha_3 \neq \beta_3$, 0 für $\alpha_3 \neq \beta_3$;
$M_3$ Matrix der Zahlen $[k_{\alpha_3\beta_3} k_{\beta_3\alpha_3}]$ ($\alpha_3, \beta_3 = 0, 1, \ldots l_2 - 1$);
$\mathfrak{G}_3 \equiv G^{(3)}, G^{(3)}_1, \ldots G^{(3)}_{l_3-1}$ die durch dasselbe Verfahren wie die Reihe $\mathfrak{G}$ aus der Matrix $M$, hergestellte Reihe von Gruppen aus der Matrix $M_3$;
$\mathfrak{H}^{(3)}_{\lambda_3}$ die Reihe derjenigen der Gruppen $H^{(2)}, H^{(2)}_1, \ldots H^{(2)}_{l_2-1}$, die wenigstens eine Zahl mit einem in den Indices der Zahlen der Gruppe $G^{(3)}_{\lambda_3}$ ($\lambda_3 = 0, 1, \ldots l_{3-1}$) vorkommenden Index enthalten;
$H^{(3)}_{\lambda_3} \equiv \mathfrak{H}^{(3)}_{\lambda_3}, G^{(3)}_{\lambda_3}$.

Die Reihe von Gruppen $\mathfrak{G}_3$ enthält dann entweder gerade nur die Gruppe $G^{(3)}$, oder mehrere Gruppen. Im ersten Falle setzen wir

$$J \equiv \mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3,$$

im zweiten Falle setzen wir das Verfahren in derselben Weise fort. Wir gelangen endlich zu einer Gruppe von Zahlen

$$J = \mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \ldots \mathfrak{G}_{u-1}$$

die das vorgelegte Problem löst.

*

Der Beweis wird durch folgende Sätze erbracht:

I. Eine Zahl $[mn]$ aus der Matrix $M$ erscheint bei beliebiger Wahl der Anfangsindices der Gruppen $F\alpha$ der Reihe $\mathfrak{F}$ in einer dieser Gruppen dann und nur dann, wenn sie entweder die kleinste der Zahlen $[m\mu]$ ($\mu \neq m$) oder der Zahlen $[n\nu]$ ($\nu \neq n$) ist.

II. Es ist möglich aus der Matrix $M$ wenigstens eine der Bedingung 1° genügende Gruppe von einander und von Null verschiedener Zahlen auszuwählen, so dass die Summe ihrer Glieder nicht grösser ist, als die Summe der Glieder irgendeiner anderen, der Bedingung 1° genügenden Gruppe von einander und von Null verschiedener Zahlen.

III. Ist $K'$ eine solche Gruppe von Zahlen, so enthält sie die Gruppe $\mathfrak{G}$.

IV. Ist $u \geqq 2$ und $v \leqq u-1$ und enthält die Gruppe $K'$ die Gruppen $\mathfrak{G}, \mathfrak{G}_1, \ldots \mathfrak{G}_{v-1}$, so enthält sie auch die Gruppe $\mathfrak{G}_v$.

V. Die Gruppe $K'$ enthält keine Zahl, die in der Gruppe $J$ nicht enthalten ist.

Brno, Jänner 1926.

22

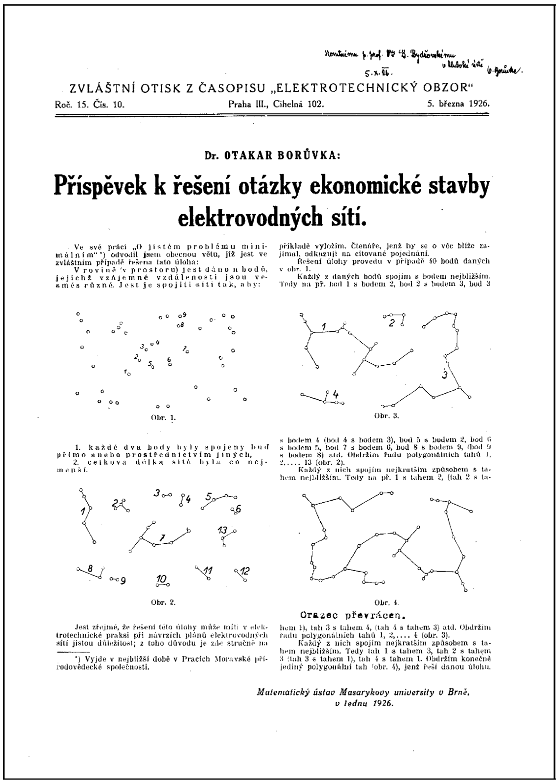
## 5. Remarks on 'Příspěvek k řešení problému ekonomické konstrukce elektrovodních sítí'

This is a strikingly different paper written in a nearly contemporary style. Example given (40 cities) is derived from the original motivation of Borůvka's research and it is the same example given at the end of [1]: the electrification of South-Moravia district in the early 20th century. (South-Moravia is one of the developed and cultured parts of Europe. It is and has been for centuries fully industrialized and yet a wine growing rich and beautiful country.)

In [39, p. 52] Borůvka clarifies how he got hold of the 'minimal problem'. The problem was communicated to him by a friend Jindřich Saxel — an employee of Západomoravské elektrárny (West-Moravian Powerplants). (Saxel as a Jew was executed in Brno by Nazis.) During the war when Czech universities were closed the company Západomoravské elektrárny offered a job to Borůvka [39, p. 83].

As well as in the translation of the first paper we tried to keep the view of the original article. A careful reader can observe that the last figure (Fig. 4) is reversed. This was noted already by Borůvka in 1926 as seen by a copy which he mailed to

Prof. Bydžovský, see the following figure).



5.x.26.

ZVLÁŠTNÍ OTISK Z ČASOPISU „ELEKTROTECHNICKÝ OBZOR“

Roč. 15. Čís. 10. Praha III., Cihelná 102. 5. března 1926.

Dr. OTAKAR BORŮVKA:

**Příspěvek k řešení otázky ekonomické stavby elektrovodných sítí.**

Ve své práci „O jistém problému minimálním“ *) odvodil jsem obecnou větu, jíž jest ve zvláštním případě řešena tato úloha:

V rovině (v prostoru) jest dáno n bodů, jejichž vzájemné vzdálenosti jsou vesměs různé. Jest je spojiti sítí tak, aby:

Obr. 1.

1. každé dva body byly spojeny buď přímo anebo prostřednictvím jiných,
2. celková délka sítě byla co nejmenší.

Obr. 2.

Jest zřejmé, že řešení této úlohy může míti v elektrotechnické praksi při návrzích plánů elektrovodných sítí jistou důležitost; z toho důvodu je zde stručně na příkladě vyložím. Čtenáře, jenž by se o věc blíže zajímal, odkazuji na citované pojednání.

Řešení úlohy provedu v případě 40 bodů daných v obr. 1.

Každý z daných bodů spojím s bodem nejbližším. Tedy na př. bod 1 s bodem 2, bod 2 s bodem 3, bod 3 s bodem 4 (bod 4 s bodem 3), bod 5 s bodem 2, bod 6 s bodem 5, bod 7 s bodem 6, bod 8 s bodem 9, (bod 9 s bodem 8) atd. Obdržím řadu polygonálních tahů 1, 2,.... 13 (obr. 2).

Obr. 3.

Každý z nich spojím nejkratším způsobem s tahem nejbližším. Tedy na př. 1 s tahem 2, (tah 2 s tahem 1), tah 3 s tahem 4, (tah 4 s tahem 3) atd. Obdržím řadu polygonálních tahů 1, 2,.... 4 (obr. 3).

Obr. 4.

Orazec převrácen.

Každý z nich spojím nejkratším způsobem s tahem nejbližším. Tedy tah 1 s tahem 3, tah 2 s tahem 3 (tah 3 s tahem 1), tah 4 s tahem 1. Obdržím konečně jediný polygonální tah (obr. 4), jenž řeší danou úlohu.

*) Vyjde v nejbližší době v Pracích Moravské přírodovědecké společnosti.

*Matematický ústav Masarykovy university v Brně, v lednu 1926.*

## 6. Modern version of Borůvka's article and algorithm

We now include a modern summary of Borůvka's article [1]:

Borůvka begins his proof by joining each vertex (= index) to its nearest neighbour. By an elaborate discussion of cases in the resulting graph-forest he gets a sequence $\mathfrak{G}$ of paths $F, F_1, \ldots, F_{i-1}$ which cover all vertices.

From this set (by an elaborate discussion) he creates the tree components. These are denoted by $G, G_1, \ldots, G_{l-1}$.

If there is only one component he gets the desired spanning tree denoted by $J$.

If there are more components he performs reduction of the matrix M to a smaller matrix $M_1$ and he explicitly remarks that the order of $M_1$ is $\leqslant n/2$.

Having established the first step Borůvka proceeds faster and by iterating the both steps he constructs sets

$$\mathfrak{G}, \mathfrak{G}_1, \ldots, \mathfrak{G}_{u-1}$$

which together form the desired set $J$.

(Thus $u$ denotes the number of iterations.)

Then he presents four statements which together establish that $J$ is the desired solution:

Theorem I claims that path covered $\mathfrak{G}$ contains an edge $[mn]$ if it is the shortest edge for either $m$ or $n$.

Theorem II claims that minimal solution exists.

Theorem III claims that any minimal solution $K'$ contains $\mathfrak{G}$.

Theorem IV claims that if a minimal solution $K'$ contains $\mathfrak{G}, \mathfrak{G}_1, \ldots, \mathfrak{G}_{v-1}$ then $K'$ contains $\mathfrak{G}_v$ as well

(and consequently $K'$ contains $J$).

Theorem V claims that a minimal solution $K'$ does not contain any edge not in $J$.

Theorem I follows from the construction (after remarks and definitions 1–6 which preceed the proof).

It is remarked in 7 that the solution set $J$ is uniquely determined.

Theorem II is quickly proved by a finiteness argument.

We know by now that Theorem III (and its iterated version Theorem IV) is the key result.

Borůvka insert Remarks 8–15 before proving Theorem III.

Thus in 8 he states that he will denote by $K'$ any solution to the MST problem and in 9 he defines a connected set $L$ (which he calls complete). In 12 he defines a path (and ordered path) joining two vertices (which he calls group) and establishes basic properties.

After this he proves Theorem III. He proceeds by contradiction. Let $[mn]$ occurs in $\mathfrak{G}$ (i.e. forest after the first iteration) and does not belong to $K'$ (a minimal tree). The key argument is short and is contained between our remarks $<\mathbf{21}>$ and $<\mathbf{24}>$: Without loss of generality, let $[mn]$ be the shortest edge incident with $m$ (by Theorem I). Thus $K'$ contains a longer edge $[mp]$. Borůvka distinguishes two cases:

(1) Every path in $K'$ from $m$ to $n$ contains the edge $[mp]$.
(2) There exists a path in $K'$ from $m$ to $n$ not containing edge $[mp]$.

In case (1) he considers the set $K'' = K' - [mp] \cup [mn]$ and proves that $K''$ is a spanning tree of shorter length.

In case (2) there exists a path in $K'$ from $m$ to $n$ which avoids $[mp]$ and thus it starts with $[m,q]$. But then $K'' = K' - [mq] \cup [mn]$ is a shorter minimal spanning tree again. This proves Theorem III.

Now Borůvka continues for four more pages to introduce elaborate constructions to handle Theorem IV. This is (in his case) necessary as he does not refer to any topology and the recursive nature of the procedure was in 1926 of course not fully understood.

Theorem V is then very short one as $J$ is the shortest solution:

‘32. Set $J$ solves the given problem’.

## 7. Borůvka's algorithm and proof in the present terminology

**Problem (MST).** Let $G=(V,E)$ be undirected connected graph with $n$ vertices and $m$ edges. For each edge $e$ let $w(e)$ be a real weight of the edge $e$ and let us assume that $w(e)\neq w(e')$ for $e\neq e'$.

Find a spanning tree $T=(V,E')$ of the graph $G$ such that the total weight $w(T)$ is minimum.

**Solution** (Borůvka's algorithm). 1. Initially all edges of $G$ are uncolored and let each vertex of $G$ be a trivial blue tree.

2. Repeat the following coloring step until there is only one blue tree.

3. *Coloring step* (Borůvka): For every blue tree $T$, select the minimum-weight uncolored edge incident to $T$. Color all selected edges blue.

**Proof** (Correctness of Borůvka's algorithm). It is easy to see that at the end of Borůvka's algorithm the blue colored edges create a spanning tree (in each step the distinct edge-weights guarantee to get a blue forest containing all vertices).

Now we show that the blue spanning tree obtained by Borůvka's algorithm is the minimum spanning tree and that is the only minimum spanning tree of the given graph $G$.

Indeed, let $T$ be a minimum spanning tree of $G$ and let $T^*$ be the blue spanning tree obtained by Borůvka's algorithm. We show that $T=T^*$:

Assume $T\neq T^*$ and let $e^*$ be the first blue colored edge of $T^*$ which does not belong to $T$. Let $P$ be the path in $T$ joining the vertices of $e^*$. It is clear that at the time when the edge $e^*$ gets blue color at least one of the edges, say $e$, of $P$ is uncolored. By the algorithm $w(e)>w(e^*)$. However then $T-e+e^*$ is a spanning tree with smaller weight, a contradiction. Thus $T=T^*$.

### Another description of Borůvka's algorithm

1. *Coloring*: For each vertex $v$ of the given graph $G$ we color blue the minimum-weight edge incident to $v$.

2. *Contraction*: We replace each blue tree by a single vertex. In this procedure we eliminate loops (i.e. edges with both ends in the same blue tree) and all the parallel edges (i.e. edges between the same pairs of blue trees) with the exception of the lowest weight edge.

3. We apply the algorithm recursively to find the blue spanning tree $T'$ of the contracted graph.

The minimum spanning tree $T$ is formed by the contracted blue edges together with the edges of $T'$.

See [12,23,44,47,50] most of the modern textbooks (such as [22,37,40]) for various descriptions of Borůvka's algorithm.

## 8. History, remarks and perspectives

The MST problem was isolated and attacked in the fifties with the vigor and confidence of then newly developing fields: theory of algorithms and computer science. The contributions were numerous and illustrious. Among others: K. Čulík [11], G. Dantzig [13], E.W. Dijkstra [14], A. Kotzig [34], J.B. Kruskal [35], H.W. Kuhn, H. Loberman, A. Weinberger, R. Kalaba [26], R.C. Prim [43], E.W. Solomon [45] (see also [38]: it is only fitting and fortunate that the recently published Borůvka's memorial volume [51] contains a reminiscence of these early days written by Kruskal [36]). These pioneering works made the MST problem popular and the further development only contributed to it. The paper of Graham and Hell [23] described accurately the development until 1985, and our paper [42] contains a historical follow up. Let us list some of the main features that indicate the role and importance of this problem in contemporary discrete mathematics along the following key words:

Complexity and Classes of Algorithms, Optimization, Relevance, Axiomatization.

**Complexity and classes of algorithms.** MST problem may be efficiently solved for large sets by several algorithms. These algorithms were studied even before the right complexity measures and problems were isolated. MST became one of the craddles of structural complexity (see the work of Edmonds in seventies, [17]). Very early attempts were made to classify the various algorithms according to their basic underlying idea (see e.g. [12,47]). Basically, all known algorithms make use of the various combinations of the following two (dual) properties of trees:

C*ut rule*: The optimal solution $T$ to MST problem contains an edge with minimal weight in every cut.

C*ircuit rule*: The edge of a circuit $C$ whose weight is larger than the weights of the remaining edges of $C$ cannot belong to the optimal solution $T$.

There is a variety of algorithms which solve MST problem efficiently. Among those the prominent role is played by Kruskal's greedy algorithm [35]. Greedy algorithm is perhaps the most thoroughly studied and used heuristic in Combinatorial Optimization. The greedy algorithm is easy to state:

**Greedy algorithm**. Sort the edges of our graph by increasing weights and then the desired set $T$ is defined recursively as follows: the next edge is added to $T$ iff together with $T$ it does not form a circuit.

**Optimization**. Let us remark that MST problem has a polynomial solution regardless of the weight function $w$ (e.g. also for negative weights). However, in the most common model (unit cost and deterministic) the complexity is still not known.

**Relevance**. Problems analogous to the MST problem were also solved efficiently, particularly the directed version of the problem (i.e. minimal branching from a given root, see [17,44]).

MST problems also appears as a subroutine to heuristic and approximate algorithms to other combinatorial optimization problems (such as Traveling Salesman Problem), see e.g. [22], [37,41,44,47].

**Axiomatization**. The class of problems solvable by Greedy Algorithm were identified with the class of matroids (no such a similar characterization seems to be known for other MST algorithms), Greedoids [32], and more recently with 'jump systems'.

While the greedy algorithm is esthetically pleasing and perhaps easiest to formulate it is NOT the fastest known algorithm (if only for the fact we need to sort the edges according to their weights that leads to a nonlinear $n \log n$ lower bound). These complexity considerations revived the interest in alternative procedures and in other algorithms for solving MST problem. It seems that this also revived the interest in the history of MST problem. And it appeared that the pre-computer age history of the problem is as illustrious as the modern development. Particularly, it appeared that the standard procedure known as Prim's algorithm [43] was discovered and formulated very clearly and concisely by the prominent number theoretician Vojtěch Jarník in 1930 [24]. (Jarník and Kössler [25] were also the first to formulate the Euclidean Steiner Tree Problem, see [33] for the history of Jarník's contribution to Combinatorial Optimization.) Consequently also the work of Otakar Borůvka was reexamined.

Borůvka formulated in [1,2] the first efficient solution of MST problem as early as 1926. His contribution was not entirely unrecognized (as opposed to Jarník's work) and both standard early references [35,43] mention Borůvka's paper. However, this reference was later dismissed as the Borůvka algorithm was regarded as 'unnecessarily complicated'. Well, perhaps a few words of explanation are in order here.

While not so easy to formulate as the greedy algorithm the Borůvka algorithm is easy to formulate as well (see Section 6).

One should stress that a concise description was not available in 1920s (not only in the pre-computer age but also in the 'pre-graph theory' age). One has to see that the operation 'contraction' became appreciated much later (in the context of planar graphs and theory of matroids) but even the term 'tree' is not mentioned in Borůvka's paper. The later seems to be the main difficulty of [1]. Instead of saying that the selected edges (in Step 1 of the algorithm) form connected components which are (obviously) trees, Borůvka elaborately constructs this tree: first he finds a maximal path $P$ containing a given point then starts with a new vertex and finds a maximal path $P'$ which either is disjoint with $P$ or terminates in a vertex of $P$ and so on. Then he combines these paths to tree-components. As a result of this Step 2 has to be tediously described and thus the description of the algorithm takes full 5 pages of [1]!

Borůvka's approach is a brute force approach par excellence. Not knowing any related literature and feeling that the problem is 'new' he arrived at the key exchange property at three different places in his article which is in the heart of all 'greedy'-type algorithms for MST. He arrived there without referring neither to cycle space and (what is now) algebraic topology (as Whitney in his pioneering work in [49]) nor to purely algebraic setting (generalizing Steinitz theorem) as Van der Waerden in [48].

He was just solving a concrete 'engineering problem' and in a strike of young genius he isolated the key property of contemporary combinational optimization.

One should regard the difficulties of the paper as technical difficulties. Moreover, there is an evidence that Borůvka had a simple description in mind as he published a follow up article in an electrotechnical journal [2] where he illustrated his method by an example (of points in the plane together with their distance as weights).

Although each of the iterations of Borůvka algorithm is more involved than the simpler rule in greedy algorithm, we need only $\log n$ of these iterations: in each step we select at least $n/2$ edges and thus the number of vertices of the contracted graph is at most half the size of the original graph. It is easy to implement the algorithm so that its complexity will be bounded by $Cm\log n$ (where $m$ is the number of edges and $C$ is a constant).

The following is another view: although we start with many (i.e. $n$) components (as many as there are blueberries in a forest; 'borůvka' is the Czech word for a 'blueberry') the number of components is halved each time and thus we are quickly done.

The 'simplicity' and effectiveness of Borůvka algorithm was recognized much later and basically during the last 10 years. Contradicting all the earlier evidence, presently it seems that Borůvka algorithm is the best algorithm available. This is based on the experimental evidence as well as its 'parallel' character and its theoretical analysis. Let us be more specific here and let us outline the recent development. It is a spectacular development as it is related to some of the key problems and advances of the modern theory of algorithms.

Given a connected undirected graph $G=(V,E)$ we denote as usual $n=|V|$ the number of its vertices and $m=|E|$ the number of its edges. As $G$ is connected it is $n-1\leqslant m$ and we can identify $m$ with the size of the input of the graph $G$. To concentrate on the combinatorial structure of the algorithms we consider the computational model unit — cost RAM with the additional restriction that the only operation allowed as the size of the weighted graph, too. This seems to be the most natural model for solving MST problem. However, one should bear in mind that the detailed complexity analysis is model-dependent as was also shown for MST e.g. in [19]. The above-mentioned algorithms are very efficient, for example the naive implementation of Greedy Algorithm is of order $mn$ (and it is easy to turn the Borůvka Algorithm into an $m\log n$ deterministic algorithm). However, this also indicates that for MST problem we can hope for very fast algorithms. Here is a summary of the results in this direction.

Yao [50] was the first to implement Borůvka Algorithm and obtained bound $m \log\log n$. This was further improved by Fredman and Tarjan [18] and finally by Gabow et al. [20,21] to the bound $m\log\beta(m,n)$ where $\beta(m,n)$ is a very slowly growing function defined as follows:

$$\beta(m,n) = \min\{i; \log\log\cdots\log(n) \leqslant m/n\}.$$

Until recently, this has been the best-known deterministic algorithm for MST problem. This algorithm also involved an important new data structure Fibonacci Heaps that found its way to standard textbooks of Theoretical Computer Science.

But one can hope for even more. For example Tarjan [46] showed that one can implement the Greedy Algorithm for graphs with presorted edge-weights so that its complexity is $m\alpha(m,n)$ where $\alpha(m,n)$ is the functional inverse to the Ackerman function. This function grows much slower than (already very slow) function $\beta$.

Very recently Bernard Chazelle succeeded to make a significant breakthrough: He devised a (presently rather complicated) deterministic algorithm for MST problem whose worst-case complexity is bounded by $Km\alpha(n)$ for a suitable constant $K$. (His work seem to cast the problems related to the function $\alpha$ in a new light.) The description of Chazelle algorithm is beyond the scope of this article, see Chazelle papers [6–9].

However fast (and 'almost' linear) the Chazelle algorithm is it is still not linear and the following seems to be the most important problem in this area:

**Problem.** Does there exist a linear deterministic algorithm which solves MST Problem? More precisely, does there exist a deterministic algorithm and a constant $C$ such that for a given weighted connected graph $G$ with $m$ edges the algorithm finds a minimum spanning tree of $G$ in at most $Cm$ steps?

One should note that many combinatorial problems can be solved by a linear deterministic algorithm (e.g. shortest path problem or finding of a planar drawing of a graph; see [47]). It is a bit surprising that this is still open for perhaps the oldest problem of Combinatorial Optimization — the MST Problem. However the problem has been intensively studied. The key role has been played by the following subproblem of MST:

**MST verification problem.** Given a weighted graph $G = (V, E)$ and its spanning tree $T$, decide whether $T$ is minimal.

Building on the early work of Tarjan [46] and an algorithm of Komlós [31] it has been showed by Dixon et al. [29] that the MST Verification problem can be solved by a linear deterministic algoritm. Recently a simpler procedure has been found by King [29]. King observed that the Komlós algorithm is simple and linear for balanced (full branched) trees. In order to apply this she transformed every tree to a full branching tree of at most double size with 'preservation' of weights. This transformation is achieved by applying the Borůvka algorithm to a tree itself, indeed King calls the tree produced in this way *Borůvka Tree*. (Borůvka tree of a tree $(V, T)$ has all the vertices as leaves and internal vertices correspond to components which appear during Borůvka algorithm, the edges represent which components produce in the next step a new component.)

This is not the end of story, perhaps rather the beginning of a new interesting period. The combination of the previously obtained methods yields unexpected results. So recently, Borůvka Algorithm has been combined with the linear verification algorithm to obtain the first linear randomized algorithm for MST problem, see Klein and Tarjan [30] and Karger et al. [27,28]. Also an optional randomized parallel algorithm has been recently found by Cole et al. [10]. See also a recent simplification by Chan [5].

In all these results the Borůvka Algorithm plays the key role. Indeed, in order to simplify their complicated parallel algorithm and its analysis Cole et al. [10] call each iteration of Borůvka Algorithm (i.e. each iteration of edge selection and subsequent contraction) *Borůvka Step*. This seems to be standard by now (see [41]).

The Combinatorial Optimization has gone a long way in its relatively short history. But it is a bit surprising how persistent are the classical motivation and algorithms. However, for a (positive) solution of some of the key problems (such as the linearity of MST problem) perhaps some new combinatorial tricks are needed.

## Appendix: Life and work of Otakar Borůvka (a brief outline)

Otakar Borůvka

Born 10.5.1899, Uherský Ostroh (Austro-Hungary, later Czechoslovakia, now Czech Republic).

| | |
|---|---|
| 1910–1916 | gymnasium (high school) Uherské Hradiště |
| 1916–1917 | military school Hranice and Mödling |
| 1918–1922 | study at the Czech Technical University, Brno |
| 1920–1921 | assistant at the Institute of Physics |
| 1920–1922 | study at the Masaryk University, Brno |
| 1921–1934 | assistant at the Masaryk University |
| 1923 | RNDr. |
| 1926–1927 | Paris |
| 1928 | docent |
| 1929–1930 | Paris (Rockefeller foundation) |
| 1930–1931 | Hamburg (supported by the Rockefeller foundation) |
| 1934 | professor at the Masaryk University (since 1946 full professor) |
| 1953 | corresponding member of the Czechoslovak Academy (ordinary member since 1965) |
| 1956 | DrSc. |
| 1959 | State Prize of Czechoslovakia |
| 1965 | founder and Editor-in-Chief of the Journal Archivum Mathematicum |
| 1969 | Dr.h.c., Bratislava |
| 1994 | Dr.h.c., Brno |
| 1995 | died in Brno (22.7.) |

Let us add at the end a few informal remarks related to [1,2]. These are works of the young mathematician, his opus No. 6, the second outside the local university journal. Borůvka was well read and well informed. The mathematical library in Brno was well stocked [39, p. 42]. One of his teacher was Matyáš Lerch, perhaps the first modern Czech mathematician who obtained the prestigeous Grand Prix de Academie de Paris in 1900, published over 230 papers and was in contact with leading mathematicians of its time (and who attended old gymnasium in Rakovník, a dear place to a subset of the authors of this article). Lerch selected Borůvka as his assistant in 1921. After a sudden death of Lerch in 1922, Borůvka became an assistant to Eduard Čech (of Stone-Čech compactification and one of the founders of topology and differential geometry). Čech directed his interest to differential geometry and arranged his stay with Elie Cartan in Paris who profoundly influenced Borůvka future mathematics. [1,2] are the only articles by Borůvka devoted to combinatorial optimization. However he was well aware of the importance of this work and in fact already during his first stay in Paris in the spring 1927 he lectured about these results in a seminar of J. L. Coolidge. He remarks that 'despite (and perhaps because of) this very unconventional topic, the lecture was received very well with an active discussion' [39, p. 59].

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