CS 6901 (Applied Algorithms) – Lecture 16

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November 13, 2014

1 Network flows

1.1 Circulations and survey design application.

Let us consider one more application, with an additional type of constraints. We will show how to reduce it to the flow networks considered before, and define a more general notion of flow, suitable for a wide range of applications.

Suppose a company wants to send a survey to some customers about some products they bought. But their requirements now have both upper and lower bounds. They want to ask each customer i at least c_i and at most c'_i questions (one customer should have at most one question for each product), and want at least p_j and at most p'_j about each product j. However, rather than asking for a maximum, they just want to know if it is possible to do, and if so, which customers should be asked about which products.

If we would only have upper bounds on the number of questions for each customer and a number of questions about each product, then we would solve it by constructing a flow network similar to the bipartite matching: make a bipartite graph with customers on one side and products on the other, each customer is connected by edges of weight 1 to the product they bought, the source s is a new vertex connected by edges of weight c'_i to customers, and products are connected to the new vertex t by edges of weight p'_j. Running Ford-Fulkerson on this network (assuming integer-valued capacities and thus integer-valued resulting flow) would tell us a maximum number of questions that could be asked over all customers. The matching between customers and the products they are asked about will correspond to edges between customers and products which got non-zero flow.

*This set of notes uses a variety of sources, in particular some material from Kleinberg-Tardos book and notes from University of Toronto CSC 364

However, this matching is not guaranteed to satisfy the lower bounds. One possibility would be to start by assigning each edge a flow equal to the lower bound. If we do that, though, then this creates an imbalance between the incoming and outgoing flow for some vertices. So some vertices become a little like sources, and some like sinks. There is a variant of flow networks, though, that deals exactly with this scenario of unbalanced flow, and feasibility rather than maximization: circulations.

In a circulation problem, there is no dedicated source or target. Instead, each vertex v has an associated demand $d_v \in \mathbb{R}$, which says how much extra flow v wants to receive (if $d_v > 0$) or give away ($d_v < 0$). And a circulation is feasible if these demands and supplies can be all satisfied, that is, there exists a flow that meets all capacity restrains (for every edge e, $0 \leq f(e) \leq c(e)$) as well as demand conditions: for each vertex v, $\Sigma_u f(u,v) - \Sigma_u(v,u) = d_v$. In particular, if there is a feasible circulation, then $\Sigma_{v,d_v<0} - d_v = \Sigma_{v,d_v>0} d_v$.

The problem of finding a feasible circulation reduces to a maximum flow problem. For that, create a new source $s*$ and new targed $t*$. We will use $s*$ to "supply extra flow" to vertices with demand $d_v > 0$, and $t*$ will "take off the extra" from the vertices with $d_v < 0$ by connecting, respectively, $s*$ to all vertices with $d_v < 0$ by edges of capacity $-d_v$, and all vertices with $d_v > 0$ to $t*$ by edges with capacity d_v. Now, if the flow that needs to be "added and then removed" is $\Sigma_{v,d_v>0} d_v$, then there is a feasible circulation in the graph.

Note that extending the circulation problem to the case where each edge has both the upper bound and the lower bound now becomes easy. Let $c(e)$ be the capacity of edge e (upper bound), as before, and $l(e)$ a lower bound on the flow on edge e. As we tried to do for the survey design problem, preset each edge with its value l_e, obtaining a network with capacities $c(e) - l(e)$ for each edge, and demands $d_v + \Sigma_u l(u,v) - \Sigma_u l(v,u)$. Now, there is a feasible circulation in this new network if there was a feasible circulation in the original network satisfying the lower bouds.

Getting back to our application to survey design, but specifying lower and upper bounds on the edges we almost obtained an instance of the circulation problem with lower bounds on edges, with all initial demands being 0. The remaining question is how to handle vertices s and t, as there are no dedicated sources/targets in the circulation, and we do not want to fix a specific demand value for them. A simple solution is to make it possible to "recirculate back" from t to s as much flow as there can be; thus, adding an edge (t,s) with capacity $\Sigma_i c'_i$ and a lower bound $\Sigma_i c_i$ completes the design of the network.