CS 6901 (Applied Algorithms) – Lecture 15

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1 Network flows

1.1 Cuts of Flow Networks

For the moment, we will concern ourselves with one more question about the algorithm. Let's assume it *does* halt; is it then the case that the flow it has found is as large as possible? The answer turns out to be *YES*! We know that if an augmenting path exists then the current flow is not optimal. We want to prove that if there is *no* augmenting path, then the current flow *is* optimal.

This is a subtle proof. Let us fix flow network $\mathcal{F} = (G, c, s, t)$, $G = (V, E)$, and flow f. We are going to introduce the new notion of a *cut* of $\mathcal{F}$. We will see that if there is no augmenting path, then there will exist a special cut that shows that f is optimal.

A *cut* (S, T) of $\mathcal{F}$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$. We define the *capacity* of (S, T) to be the sum of the capacities over all edges going from S to T; note that this is a sum of nonnegative numbers. We define the *flow* across (S, T) to be the sum of the flows over all edges going from S to T; note that this sum may consist of negative numbers. More formally:

The *capacity* of the cut (S, T) is defined by

$$c(S,T) = \sum_{(x,y) \in (S \times T) \cap E} c(x,y)$$

*This set of notes uses a variety of sources, in particular some material from Kleinberg-Tardos book and notes from University of Toronto CSC 364

The *flow* across (S, T) is

$$f(S,T) = \sum_{(x,y)\in(S\times T)\cap E} f(x,y) - f(y,x)$$

Example: Consider our earlier example of the flow network $\mathcal{F}$ with flow f'. Consider the cut $(S,T) = (\{s, v_3\}, \{t, v_1, v_2, v_4\})$. We have $c(S,T) = 20 + 13 + 8 + 22 = 63$ and $f'(S,T) = 13 + 12 + (-14) + (-7) + 21 = 25$.

We see that $f(S,T)$ in the above example is exactly equal to $|f|$, and this is no coincidence. Intuitively it makes sense that the amount flowing out of s should be exactly the same as the amount flowing across any cut, and this is proven in the next lemma. In particular, by considering the cut $(V - \{t\}, \{t\})$, we see that $|f|$ is exactly equal to the amount flowing into t.

Lemma 1 *Fix flow network $\mathcal{F} = (G, c, s, t)$ and flow f. Then for every cut (S,T), $f(S,T) = |f|$.*

Corollary 1 *Fix flow network $\mathcal{F} = (G, c, s, t)$ and flow f. Then for every cut (S,T), $f(S,T) \leq c(S,T)$.*

Corollary 2 *The value of every flow in $\mathcal{F}$ is less than or equal to the capacity of every cut of $\mathcal{F}$.*

We now state and prove the famous "max-flow, min cut" theorem. This theorem says that the maximum value over all flows in $\mathcal{F}$ is exactly equal to the minimum capacity over all cuts. It also tells us that if $\mathcal{F}$ has no augmenting paths with respect to a flow f, then $|f|$ is the maximum possible.

Theorem 1 *(MAX-FLOW, MIN-CUT THEOREM)*
Fix flow network $\mathcal{F} = (G, c, s, t)$, $G = (V, E)$, and flow f. Then the following are equivalent

1) f is a max flow (that is, a flow of maximum possible value) in $\mathcal{F}$.

2) There are no augmenting paths with respect to f.

3) $|f| = c(S,T)$ for some cut (S,T) of $\mathcal{F}$.

Proof:
(1) $\Rightarrow$ (2)
Suppose (1) holds. We have already seen that if there were an augmenting path with respect

to f, then a flow with value larger than $|f|$ could be constructed. Since f is a max flow, there must be no augmenting paths.

(2) $\Rightarrow$ (3)
Suppose (2) holds. Then there is no path from s to t in G_f.
Let $S = \{v \in V \,|\, \text{there exists a path from } s \text{ to } v \text{ in } G_f\}$, and let $T = V - S$. Clearly (S, T) is a cut. We claim that $|f| = c(S, T)$. ¿From the above Lemma 3, it suffices to show that $f(S, T) = c(S, T)$. For this, it suffices to show that for every edge $(u, v) \in (S \times T) \cap E$, $f(u, v) = c(u, v)$. So consider such an edge (u, v). If we had $f(u, v) < c(u, v)$, then (u, v) would be an edge with positive residual capacity, and hence (u, v) would be an edge of G_f, and hence (since $u \in S$), there would be a path in G_f from s to v, and hence $v \in S$ – a contradiction.

(3) $\Rightarrow$ (1)
Suppose (3) holds. Let (S, T) be a cut of $\mathcal{F}$ such that $|f| = c(S, T)$. ¿From the above corollary, we know that every flow has value less than or equal to $c(S, T)$, and hence every flow has value less than or equal to $|f|$. So f is a max flow. □

This theorem tells us that if Ford-Fulkerson halts, then the resulting flow is optimal.

1.2 Open-pit mining/project selection application

Recall our second motivating example: given a mineral deposit where the cost and profit of excavating of every cubic meter of soil is known, and a cubic meter of soil can be excavated only if the one directly above is taken out, determine which ones to take out when constructing an open-pit mine. That is, we want to find a "cut" in the ground which will give us the best profit.

Construct a weight graph G as follows. Make a vertex for every cubic meter of soil (call them units, for brevity), and add two extra vertices s and t. For unit, make an edge of cost ∞ from its vertex to the vertex corresponding to the unit right above it. Connect s to every vertex corresponding to units with $cost < profit$ by an edge of weight $p_i = profit_i - cost_i$. Connect every vertex corresponding to a unit with $cost > profit$ (that is, net profit $p_i < 0$) to t by an edge of weight $-p_i = cost_i - profit_i$.

Now, run Ford-Fulkerson, and calculate the minimal cut by running BFS from s in the final residual network to compute the set of vertices S reachable from s. Now, this set of vertices (minus s) is the set of units that should be dug out to maximize the profit.