1 Ladner’s theorem

Theorem 1 (Ladner’s Theorem). If $P \neq NP$, then there are problems in $NP$ that are neither in $P$ nor $NP$-complete.

Thus, in general, there must be problems in $NP$ that are not in $P$ and not $NP$-complete (unless $NP = P$ and then everything in $NP$ is in $P$). Are there any natural candidates? Probably the best candidate is the problem Graph Isomorphism, which asks if two given graphs are isomorphic. It is in $NP$ since we can nondeterministically guess an isomorphism and then deterministically check in polytime if it is indeed an isomorphism. It is open if the problem is in $P$ (except for some special cases); it is not known if the problem can be solved by a polytime quantum algorithm, despite considerable interest to this question from the quantum complexity community.

On the other hand, there is some evidence that Graph Isomorphism cannot be $NP$-complete. If it is, then something unlikely will happen: the polytime hierarchy would collapse.

2 Schaefer’s theorem

Whereas Ladner’s theorem says that if $P \neq NP$ then there are problems that are neither $NP$-complete nor polytime-solvable, Schaefer’s theorem states that classes of problems obtained by very natural type of restrictions on SAT are either in $P$ or $NP$-complete. More precisely, let us differentiate SAT problems by the types of clauses occurring in them. So far, we have seen two subtypes of SAT: Horn formulae where every clause has at most one positive literal (that was $P$-complete), and 2CNF where each clause has at most three variables.

Theorem 2 (Schaefer’s dichotomy theorem). Any satisfiability problem restricted by the type of clauses in $CNF$ is either trivially solvable or complete for one of the following complexity classes under logspace reductions:

1. $L(=SL)$, if the clauses are of the form $(x \oplus y), (\neg x \oplus y)$.
2. $NL$, if the clauses are of the form $(x \lor y), (\neg x \lor y), (\neg x \lor \neg y)$.
3. $\oplus L$, if the clauses are of the form $(x \oplus y \oplus z \oplus \ldots), (\neg x \oplus y \oplus z \oplus \ldots)$.
4. $P$, if either all clauses are Horn or all clauses are dual horn (at most one negative literal per clause).
5. $NP$-complete, otherwise.