1 Ladner's theorem

Theorem 1 (Ladner's Theorem)**.** *If* $\mathsf{P} \neq \mathsf{NP}$*, then there are problems in* NP *that are neither in* P *nor* NP*-complete.*

Thus, in general, there must be problems in NP that are not in P and not NP-complete (unless $\mathsf{NP} = \mathsf{P}$ and then everything in NP is in P). Are there any natural candidates? Probably the best candidate is the problem Graph Isomorphism, which asks if two given graphs are isomorphic. It is in NP since we can nondeterministically guess an isomorphism and then deterministically check in polytime if it is indeed an isomorphism. It is open if the problem is in P (except for some special cases); it is not known if the problem can be solved by a polytime quantum algorithm, despite considerable interest to this question from the quantum complexity community.

On the other hand, there is some evidence that Graph Isomorphism cannot be NP-complete. If it is, then something unlikely will happen: the polytime hierarchy would collapse.

2 Schaefer's theorem

Whereas Ladner's theorem says that if $P \neq NP$ then there are problems that are neither NP-complete nor polytime-solvable, Schaefer's theorem states that classes of problems obtained by very natural type of restrictions on SAT are either in P or NP-complete. More precisely, let us differentiate SAT problems by the types of clauses occurring in them. So far, we have seen two subtypes of SAT: Horn formulae where every clause has at most one positive literal (that was P-complete), and 2CNF where each clause has at most three variables.

Theorem 2 (Schaefer's dichotomy theorem)**.** *Any satisfiability problem restricted by the type of clauses in CNF is either trivially solvable or complete for one of the following complexity classes under logspace reductions:*

1. *L(=SL), if the clauses are of the form* $(x \oplus y), (\neg x \oplus y)$.
2. *NL, if the clauses are of the form* $(x \vee y), (\neg x \vee y), (\neg x \vee \neg y)$.
3. $\oplus$*L, if the clauses are of the form* $(x \oplus y \oplus z \oplus \dots), (\neg x \oplus y \oplus z \oplus \dots)$.
4. *P, if either all clauses are Horn or all clauses are dual horn (at most one negative literal per clause).*
5. *NP-complete, otherwise.*