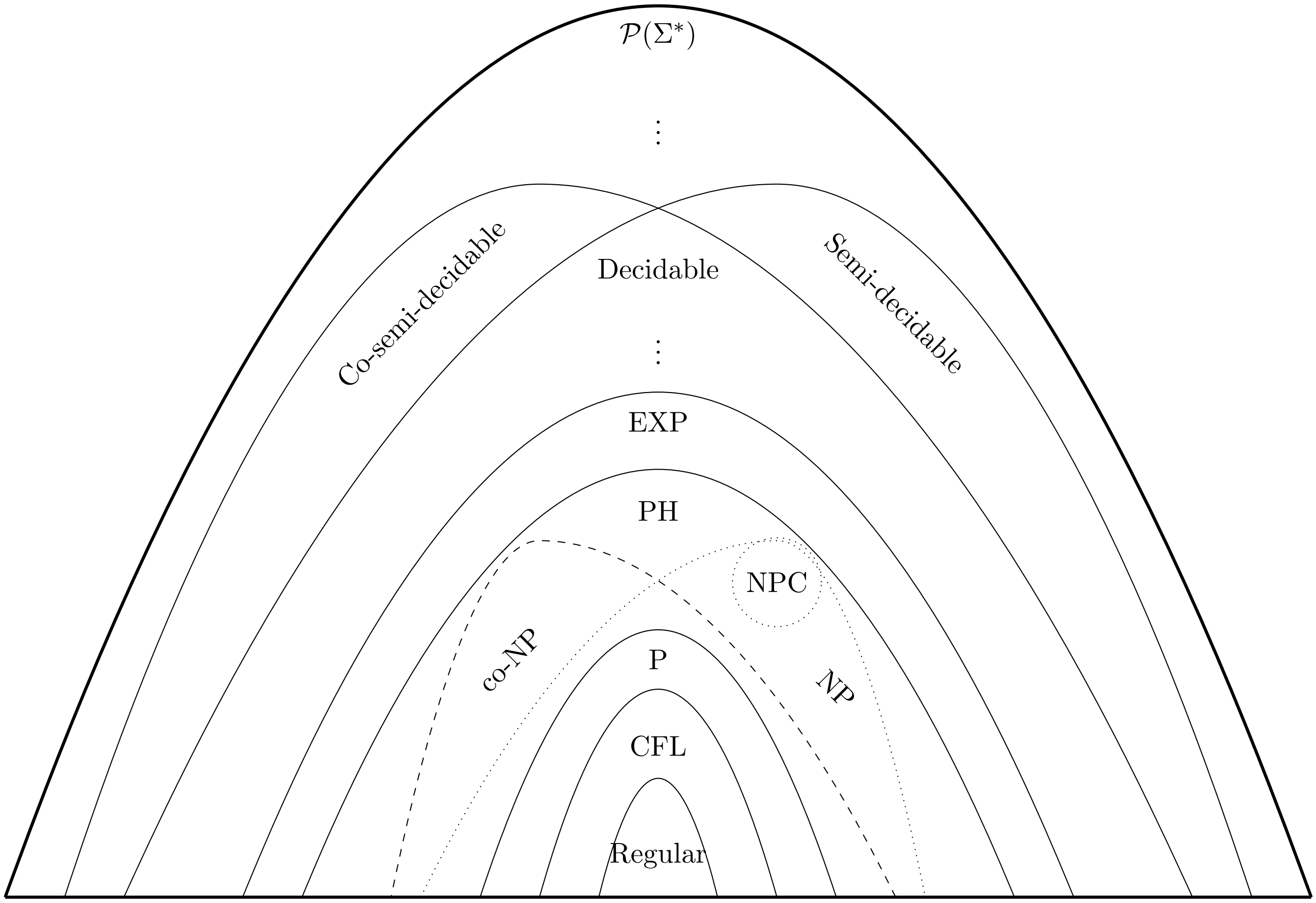
Exam study sheet for CS3719



Regular languages and finite automata

- An *alphabet* is a finite set of symbols. Set of all finite strings over an alphabet Σ is denoted Σ^*. A *language* is a subset of Σ^*. Empty string is called ϵ (epsilon).
- *Regular expressions* are built recursively starting from $\emptyset, \epsilon$ and symbols from Σ and closing under Union ($R_1 \cup R_2$), Concatenation ($R_1 \circ R_2$) and Kleene Star (R^* denoting 0 or more repetitions of R) operations. These three operations are called regular operations.
- A Deterministic Finite Automaton (DFA) D is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is the alphabet, $\delta : Q \times \Sigma \to Q$ is the transition function, q_0 is the start state, and F is the set of accept states. A DFA accepts a string if there exists a sequence of states starting with $r_0 = q_0$ and ending with $r_n \in F$ such that $\forall i, 0 \leq i < n, \delta(r_i, w_i) = r_{i+1}$. The language of a DFA, denoted $\mathcal{L}(D)$ is the set of all and only strings that D accepts.
- A language is called *regular* iff it is recognized by some DFA.
- **Theorem:** The class of regular languages is closed under union, concatenation and Kleene star operations.

- A *non-deterministic* finite automaton (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where Q, Σ, q_0 and F are as in the case of DFA, but the transition function δ is $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathcal{P}(Q)$. Here, $\mathcal{P}(Q)$ is the powerset (set of all subsets) of Q. A non-deterministic finite automaton *accepts* a string $w = w_1 \dots w_m$ if there exists a sequence of states $r_0, \dots r_m$ such that $r_0 = q_0$, $r_m \in F$ and $\forall i, 0 \leq i < m, r_{i+1} \in \delta(r_i, w_i)$.

- **Theorem:** For every NFA there is a DFA recognizing the same language. The construction sets states of the DFA to be the powerset of states of NFA, and makes a (single) transition from every set of states to a set of states accessible from it in one step on a letter following with all states reachable by (a path of) ϵ-transitions. The start state of the DFA is the set of all states reachable from q_0 by following possibly multiple ϵ-transitions.

- **Theorem:** A language is recognized by a DFA if and only if it is generated by some regular expression. In the proof, the construction of DFA from a regular expression follows the closure proofs and recursive definition of the regular expression. The construction of a regular expression from a DFA first converts DFA into a Generalized NFA with regular expressions on the transitions, a single distinct accept state and transitions (possibly $\emptyset$) between every two states. The proof proceeds inductively eliminating states until only the start and accept states are left.

- **Lemma** The *pumping lemma for regular languages* states that for every regular language A there is a pumping length p such that $\forall s \in A$, if $|s| > p$ then $s = xyz$ such that 1) $\forall i \geq 0, xy^iz \in A$. 2) $|y| > 0$ 3) $|xy| < p$. The proof proceeds by setting p to be the number of states of a DFA recognizing A, and showing how to eliminate or add the loops. This lemma is used to show that languages such as $\{0^n1^n\}, \{ww^r\}$ and so on are not regular.

Context-free languages and Pushdown automata.

- A *pushdown automaton* (PDA) is a "NFA with a stack"; more formally, a PDA is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the set of states, Σ the input alphabet, Γ the stack alphabet, q_0 the start state, F is the set of finite states and the transition function $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \to \mathcal{P}(Q \times (\Gamma \cup \{\epsilon\}))$.

- A *context-free grammar* (CFG) is a 4-tuple (V, Σ, R, S), where V is a finite set of variables, with $S \in V$ the start variable, Σ is a finite set of `terminals` (disjoint from the set of variables), and R is a finite set of rules, with each rule consisting of a variable followed by $->$ followed by a string of variables and terminals.

- Let $A \to w$ be a rule of the grammar, where w is a string of variables and terminals. Then A can be replaced in another rule by w: uAv in a body of another rule can be replaced by uwv (we say uAv yields uwv,denoted $uAv \Rightarrow uwv$). If there is a sequence $u = u_1, u_2, \dots u_k = v$ such that for all i, $1 \leq i < k$, $u_i \Rightarrow u_{i+1}$ then we say that u derives v (denoted $v \overset{*}{\Rightarrow} v$.) If G is a context-free grammar, then the language of G is the set of all strings of terminals that can be generated from the start variable: $\mathcal{L}(G) = \{w \in \Sigma^* | S \overset{*}{\Rightarrow} w\}$. A *parse tree* of a string is a tree representation of a sequence of derivations; it is *leftmost* if at every step the first variable from the left was substituted. A grammar is called *ambiguous* if there is a string in a grammar with two different (leftmost) parse trees.

- A language is called a *context-free language* (CFL) if there exists a CFG generating it.

- **Theorem** Every regular language is context-free.

- **Theorem** A language is context-free iff some pushdown automaton recognizes it. The proof of one direction constructs a PDA from the grammar (by having a middle state with "loops" on rules; loops consist of as many states as needed to place all symbols in the rule on the stack).

- **Lemma** The *pumping lemma for context-free languages* states that for every CFL A there is a pumping length p such that $\forall s \in A$, if $|s| > p$ then $s = uvxyz$ such that 1) $\forall i \geq 0, uv^ixy^iz \in A$. 2) $|vy| > 0$ 3) $|vxy| < p$. This lemma is used to show that languages such as $\{a^nb^nc^n\}, \{ww\}$ and so on are not regular.

- **Theorem** The class of CFLs is *not* closed under complementation and intersection (although it is closed under union, Kleene star and concatenation).

- **Theorem** There are context-free languages not recognized by any deterministic PDA.

Turing machines and decidability.

- A Turing machine is a finite automaton plus an infinite read/write memory (tape). Formally, a Turing machine is a 6-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$. Here, Q is a finite set of states as before, with three special states q_0 (start state), q_{accept} and q_{reject}. The last two are called the halting states, and they cannot be equal. Σ is a finite input alphabet. Γ is a tape alphabet which includes all symbols from Σ and a special symbol for blank, $\sqcup$. Finally, the transition function is $\delta : Q. \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ where L, R mean move left or right one step on the tape. Also know encoding languages and Turing machines as binary strings.

- Equivalent (not necessarily efficiently) variants of Turing machines: two-way vs. one-way infinite tape, multi-tape, non-deterministic.

- *Church-Turing Thesis* Anything computable by an algorithm of any kind (our intuitive notion of algorithm) is computable by a Turing machine.

- A Turing machine M *accepts* a string w if there is an accepting computation of M on w, that is, there is a sequence of configurations (state,non-blank memory,head position) starting from q_0w and ending in a configuration containing q_{accept}, with every configuration in the sequence resulting from a previous one by a transition in δ of M. A Turing machine M *recognizes* a language L if it accepts all and only strings in L: that is, $\forall x \in \Sigma^*$, M accepts x iff $x \in L$. As before, we write $\mathcal{L}(M)$ for the language accepted by M.

- A language L is called *Turing-recognizable* (also *recursively enumerable, r.e*, or *semi-decidable*) if $\exists$ a Turing machine M such that $\mathcal{L}(M) = L$. A language L is called *decidable* (or *recursive*) if $\exists$ a Turing machine M such that $\mathcal{L}(M) = L$, and additionally, M halts on all inputs $x \in \Sigma^*$. That is, on every string M either enters the state q_{accept} or q_{reject} in some point in computation. A language is called *co-semi-decidable* if its complement is semi-decidable.

- Semi-decidable languages can be described using unbounded $\exists$ quantifier over a decidable relation; co-semi-decidable using unbounded $\forall$ quantifier. There are languages that are higher in the arithmetic hierarchy than semi- and co-semi-decidable; they are described using mixture of $\exists$ and $\forall$ quantifiers; the number of alternations of quantifiers is the level in the hierarchy. In particular, the decidable predicate can be $V_A(M, w, y)$ which is true iff y encodes an accepting computation of M on w. V_R and V_H are defined similarly for y a rejecting and a halting computation, respectively.

- If a language is both semi-decidable and co-semi-decidable, then it is decidable.

- Universal language $A_{TM} = \{\langle M, w\rangle \mid w \in \mathcal{L}(M)\} = \{\langle M, w\rangle \mid \exists y V_A(M, w, y)\}$. A_{TM} is undecidable: proof by contradiction. Examples of undecidable languages: A_{TM}, $Halt_B$, NE (semi-decidable), *Empty* (co-semi-decidable), $L = \{\langle M_1, w_1, M_2, w_2\rangle \mid w_1 \in \mathcal{L}(M_1) \text{ and } w_2 \notin \mathcal{L}(M_2)\}$ *Total* (neither), three languages from the assignment.

- A *many-one* reduction: $A \leq_m B$ if exists a computable function f such that $\forall x \in \Sigma_A^*$, $x \in A \iff f(x) \in B$. To prove that B is undecidable, (not semi-decidable, not co-semi-decidable) pick A which is undecidable (not semi, not co-semi.) and reduce A to B. To prove that a language L is in a class (e.g., semi-decidable), give an algorithm (e.g, M_L).

Complexity theory, NP-completeness

- A Turing machine M runs in time $t(n)$ if for any input of length n the number of steps of M is at most $t(n)$ (worst-case running time).

- A language L is in the complexity class $\mathtt{P}$ (stands for *Polynomial time*) if there exists a Turing machine M, $\mathcal{L}(M) = L$ and M runs in time $O(n^c)$ for some fixed constant c. The class $\mathtt{P}$ is believed to capture the notion of efficient algorithms.

- A language L is in the class $\mathtt{NP}$ if there exists a *polynomial-time verifier*, that is, a relation $R(x, y)$ computable in polynomial time such that $\forall x, x \in L \iff \exists y, |y| \leq c|x|^d \wedge R(x, y)$. Here, c and d are fixed constants, specific for the language.

- A different, equivalent, definition of $\mathtt{NP}$ is a class of languages accepted by polynomial-time *non-deterministic* Turing machines. The name $\mathtt{NP}$ stands for "Non-deterministic Polynomial-time".

- $\mathtt{P} \subseteq \mathtt{NP} \subseteq \mathtt{EXP}$, where $\mathtt{EXP}$ is the class of languages computable in time exponential in the length of the input. It is known that $\mathtt{P} \subsetneq \mathtt{EXP}$. All of them are decidable. Alternating quantifiers, get polynomial-time hierarchy $\mathtt{PH}$: $\mathtt{P} \subseteq \mathtt{NP} \cap \mathtt{coNP} \subseteq \mathtt{NP} \cup \mathtt{coNP} \subseteq \mathtt{PH} \subseteq \mathtt{PSPACE} \subseteq \mathtt{EXP}$.

- Examples of languages in $\mathtt{P}$: connected graphs, relatively prime pairs of numbers (and, quite recently, prime numbers), palindromes,etc.

- Examples of languages in $\mathtt{NP}$: all languages in P, Clique, Hamiltonian Path, SAT, etc. Technically, functions computing an output other than yes/no are not in $\mathtt{NP}$ since they are not languages.

- Examples of languages not known to be in $\mathtt{NP}$: LargestClique, TrueQuantifiedBooleanFormulas.

- Major Open Problem: is $\mathtt{P} = \mathtt{NP}$? Widely believed that not, weird consequences if they were, including breaking all modern cryptography and automating creativity.

- If $\mathtt{P} = \mathtt{NP}$, then can compute witness y in polynomial time. Same idea as search-to-decision reductions.

- *Polynomial-time reducibility*: $A \leq_p B$ if there exists a *polynomial-time computable* function f such that $\forall x \in \Sigma, x \in A \iff f(x) \in B$.

- A language L is $\mathtt{N}$-hard if every language in $\mathtt{NP}$ reduces to L. A language is $\mathtt{NP}$-complete it is both in $\mathtt{NP}$ and $\mathtt{NP}$-hard.

- Cook-Levin Theorem states that SAT is $\mathtt{NP}$-complete. The rest of $\mathtt{NP}$-completeness proofs we saw are by reducing SAT (3SAT) to the other problems (also mentioned a direct proof for CircuitSAT in the notes).

- Examples of $\mathtt{NP}$-complete problems with the reduction chain:
  - $SAT \leq_p 3SAT$
  - $3SAT \leq_p IndSet \leq_p Clique$
  - $Partition \leq_p SubsetSum \leq_p GKP$ (skipped $3SAT \leq SubsetSum$; see the book.)

Algorithm design and analysis

- **Greedy algorithms (chapter 12-13)** Sort items then go through them either picking or ignoring each; never reverse a decision. Running time usually $O(n \log n)$ where n is the number of elements (depends on data structures used, too). Often does not work or only gives an approximation; when it works, correctness proof by induction on the number of steps (i.e., S_i is the solution set after considering i^{th} element in order.) Examples of greedy algorithms: 2-approximation for Simple Knapsack, Fractional knapsack, Kruskal's algorithm for Minimal Spanning Tree, Dijkstra's algorithm, scheduling with deadlines and profits.

- **Dynamic programming (chapter 12-13)** Precompute partial solutions starting from the base cases, keep them in a table, compute the table from already precomputed cells (e.g., row by row, but can be different). Arrays can be 1,2, 3-dimensional (possibly more), depends on the problem. Running time a function of the size of the array – might be not polynomial (e.g., scheduling with very large deadlines)! Examples: Scheduling, Knapsack, Longest Common Subsequence, Longest Increasing Subsequence, All Pair Shortest Path (Floyd-Warshall). Steps of design:

  1. Define an array; that is, state what are the values being put in the cells, then what are the dimensions and where the value of the best solution is stored. E.g.: $A(i,t)$ stores the profit of the best schedule for jobs from 1 to i finishing by time t, where $1 \leq i \leq n$, and $0 \leq t \leq maxd_i$. Final answer value is $A(n, maxd_i)$.
  2. Give a recurrence to compute A from the previous cells in the array, including initialization. E.g. (longest common subsequence) $A(i,j) = \begin{cases} A(i-1,j-1)+1 & x_i = y_j \\ \max\{A(i-1,j), A(i,j-1)\} & \text{otherwise} \end{cases}$
  3. Give pseudocode to compute the array (usually we omitted it in class).
  4. Explain how to recover the actual solution from the array (usually using a recursive $PrintOpt()$ procedure to retrace decisions).

- Dynamic programming does not necessarily produces polynomial-time algorithms (e.g. Scheduling, Knapsack, SubsetSum, where the value of a binary variable is an array dimension).

- For some problem, dynamic programming allows to get arbitrary good approximation (trading off array size vs. accuracy).

- **Backtracking** Used when others don't work; usually exponential time, but faster than testing all possibilities. Make a decision tree of possibilities, go through the tree recursively, if some possibilities fail, backtrack.

- **Network flows**

  - *Flow network*: a directed graph with (non-negative) weights on edges called *capacities* c_e, and two special vertices *source* s and *sink* t.
  - A *flow* in a flow network is a function $f \colon E \to \mathbb{R}^+$ from edges to numbers satisfying
    1. *Capacity constraints*: For every edge $e \in E$ $0 \leq f(e) \leq c_e$. That is, flow on an edge never exceeds the edge's capacity.
    2. *Flow conservation*: For every vertes v other than s and t, $\Sigma_{e \text{ into } v} f(e) = \Sigma_{e \text{ out of } v} f(e)$. That is, for all intermediate vertices everything that goes into v has to get out.
  - A *cut* in a flow network is partition of vertices into a set S containing s, and a set T containing t. The capacity of a cut is the sum of capacities of all edges from vertices in S to vertices in T.
  - *Min cut max flow theorem*: The maximum possible flow in a flow network equals capacity of the smallest-capacity cut.

- Given a flow network G with some flow f, the *residual network* G_f contains 1) all edges of G with remaining capacity $c_e - f(e) > 0$, with capacities $c_e - f(e)$. 2) For each edge (u, v) with flow $f(e) > 0$, a backward edge (v, u) with capacity $f(e)$.
- An *augmenting path* is a path from s to t in the residual network. Its capacity is the minimum capacity over edges on the path.
- *Ford-Fulkerson-Edmonds-Karp algorithm*: Start with the original flow network as residual network. While t is reachable from s in the residual network, select the shortest augmenting path, update the residual network with flow along this augmenting path equal to its capacity.
- When there are no more augmenting paths, a minimum-capacity cut consists of $S =$ all vertices reachable from s, T the rest.
- To solve maximum matching in a bipartite graph problem, add new vertices s and t to the graph, with edges from s to one side of the bipartite graph and from the other side to t. Make all capacities 1. Compute the max. flow. Now, all edges between two sides of the graph with non-zero flow form a maximum matching.