CS 2742 (Logic in Computer Science)
Lecture 4

Antonina Kolokolova

January 17, 2014

1.1 Simplifying propositional formulas.

In the last class, we talked about logical equivalences, and listed a few most notable logical identities. Now we can apply these identities to simplify propositional formulas.

Example 1.

$$\begin{aligned} & (p \wedge q) \vee \neg(\neg p \vee \neg q) & \\ \iff & (p \wedge q) \vee (\neg\neg p \wedge \neg\neg q) & \text{Apply DeMorgan's} \\ \iff & (p \wedge q) \vee (p \wedge q) & \text{Double Negation (twice)} \\ \iff & (p \wedge q) & \text{Idempotence} \end{aligned}$$

Notice that the logic identities are stated only for the logical connectives $\wedge, \vee, \neg$. In order to deal with $\rightarrow$ and $\iff$ we use their definitions: for example, $A \rightarrow B$ becomes $\neg A \vee B$.

Here we go over a few more examples similar to your assignment, simplifying formulas until they are as small as we can (reasonably easily) get using the rules from the last lecture. For the assignment question on negation, you would mainly apply DeMorgan and Double Negation to a negated formula, which is a much simpler procedure.

Example 2.

$$
\begin{aligned}
& p \leftrightarrow ((q \wedge \neg r) \rightarrow q) & \\
\iff & p \leftrightarrow (\neg(q \wedge \neg r) \vee q) & \text{Definition of } \rightarrow \\
\iff & p \leftrightarrow ((\neg q \vee \neg\neg r) \vee q) & \text{DeMorgan} \\
\iff & p \leftrightarrow ((\neg\neg r \vee \neg q) \vee q) & \text{Commutativity} \\
\iff & p \leftrightarrow (\neg\neg r \vee (\neg q \vee q)) & \text{Associativity (dropping parentheses)} \\
\iff & p \leftrightarrow (\neg\neg r \vee T) & \text{Definition of T} \\
\iff & p \leftrightarrow T & \text{Identity} \\
\iff & p & \text{Because } \leftrightarrow \text{ is an equivalence}
\end{aligned}
$$

The last step could be done more formally as follows:

$$
\begin{aligned}
& p \leftrightarrow T & \\
\iff & (p \rightarrow T) \wedge (T \rightarrow p) & \text{Definition of } \leftrightarrow \\
\iff & (\neg p \vee T) \wedge (\neg T \vee p) & \text{Definition of } \rightarrow \\
\iff & (\neg p \vee T) \wedge (F \vee p) & \text{Definition of } F \\
\iff & T \wedge (F \vee p) & \text{Identity} \\
\iff & (F \vee p) & \text{Identity} \\
\iff & p & \text{Identity}
\end{aligned}
$$

1.2 Conditional statements

Here is a puzzle that I was planning to give you in the last lecture.

Puzzle 3 (Wason Selection Task)**.** You see 4 cards on the table; all cards have a letter on one side and a number on the other. On those four cards you see written “A”, “D”, “4” and “7”. Which cards do you need to turn over to verify the statement “If a card has a vowel on one side, then it has an even number on the other”?

Pretty much everybody immediately sees that a card with “A” needs to be turned over to verify that it has an even number on the other side. But for the other cards, many people would say “4”. However, turning “4” over does not give us much information about the statement: whether it has a vowel or a consonant, it still does not prove or disprove the statement: there are no restrictions on cards with consonants on them. To see which card to turn over, think of a *contrapositive* statement: “if a card does not have an even number on one side, then it does not have a vowel on the other”. So the card that has to be checked is the card with a “7”.

This puzzle gives an example of a conditional statement, that is, a statement of the form "if A then B", $A \to B$. Recall that we logically define $p \to q \iff (\neg p \vee q)$. Here is another example of a conditional statement. Note that its negation is *not* a conditional statement itself, but rather an "and": an implication is false in only one situation, when A is true and B is false, and the negation of the implication states that it is indeed the case. Alternatively, you can verify it by applying DeMorgan's law and double negations to the $\neg(\neg A \vee B)$ formula.

Example 3. If Jane is in London, then she is in England.
Negation: Jane is in London, and she is not in England (e.g., London, Ontario).

We use the following terminology when talking about conditional statements:

1) *Contrapositive* of $p \to q$ is $\neg q \to \neg p$. True whenever the original implication is.

   Proof. Recall that $(p \to q) \iff (\neg p \vee q)$. Now,

$$\begin{aligned} & \neg q \to \neg p \\ \iff & (\neg\neg q \vee \neg p) && \text{Definition of } \to \\ \iff & (q \vee \neg p) && \text{Double negation} \\ \iff & (\neg p \vee q) && \text{Commutativity } \iff (p \to q) \qquad \text{Definition of } \to \end{aligned}$$

   Thus, a contrapositive of an if-then statement is logically equivalent to the original statement. □

   In the cards example, the contrapositive is "if a card does not have an even number on one side, then it does not have a vowel on the other side". Thus, cards with odd number facing up need to be checked.

2) *Converse* $q \to p$ and *inverse* $\neg p \to \neg q$. Contrapositives of each other, can have a different truth value from $p \to q$.

   So a converse in the cards example would be "if a card has an even number on one side, then it has a vowel on the other". You can see that this does not have the same truth value as the original statement: the first one is true when the card with 4 has a consonant (say, B) on the other side, but the converse would be falsified by this scenario.

3) *Sufficient* condition: p is sufficient for q if $p \to q$. The "if" part of if-then.
   Neccessary condition: q is necessary for p if $\neg q \to \neg p$, that is, $p \to q$. The "then" part of "if-then".

   So if we know that the puzzle statement is true for the 4 given cards, then it is sufficient to know that a card has a vowel (say A) on one side to conclude that the other side has an even number. And, if a card has say A on one side, it is necessary for it to have an even number on the other for the whole "if card has a vowel on one side then it has an even number on the other" to be true.

4) If and only if (p iff q, $p \leftrightarrow q$) means $(p \to q) \wedge (q \to p)$.

In many daily situations people use "if.. then.." construct implying a biconditional (if and only if), and sometimes even the "only if" direction. For example, a parent telling a kid "if you eat your veggies then you'll get to eat your desert" probably means the converse – if the kid doesn't eat her veggies, then she'd be punished by not getting her desert. In that case, the context would allow us to interpret the sentence the way it was intended (although a logic-savvy kid would catch her dad on that). But in sciences, medicine, law where it is very important to avoid ambiguity; there, "if.. then" statements should be interpreted according to the rules of logic.

In a book "Logic made easy" by Deborah Bennet (where she also talks about the Wason Selection test and the puzzle I will give at the end of the class) she mentions a common mistake doctors made when interpeting the meaning of sensitivity of a test (probability of a correct positive answer). The sensitivity is often stated as in "if a person is sick, then the probability of the test being positive is 90%". Now suppose somebody tested positive. What is the probability that they are indeed sick? It is tempting to say "90%"; however this number is irrelevant here: it could have been a test that often returns positive even for healthy people.

For example, consider airport security. Suppose somebody says that if a person is a terrorist carrying a weapon, then the metal detector rings with probability 90%. But this is definitely not the same as saying that every time the detector rings, with 90% chance it is a terrorist: much more likely a person forgot to take keys or coins out of their pocket.

You can see how this can come up in e.g. American court system a lot: saying "if a subject committed a crime, then his fingerprint test will come up positive with 99% probability, and it did come up positive" will make jurors think that the accused really committed the crime. If they know logic, though, they will ask: "but what is the probability that a test will be positive for an innocent person"?

Let me finish with a version of another puzzle from the same book.

Puzzle 4 (Colours and shapes)**.** Suppose you meet a parent with a little baby girl; the parent tells you that she is attracted to some colours and some shapes, and goes for a toy which has either the colour or the shape she likes (or both, of course). In her play area, you see several plush toys: a blue square, a blue circle, a yellow square and a yellow circle. You see the baby reaching for a blue circle. What, if anything, can you infer about her liking or not liking other three toys?