CS 2742 (Logic in Computer Science) – Winter 2014
Lecture 20

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Recall the puzzle from last class about a proof that all horses are white:

Puzzle 1. What is wrong with the following induction proof of: "All horses are white"?

1) Let $P(n)$ be: any n horses are white.

2) Base case: 0 horses are white.

3) Ind. hyp.: if any set of k horses are white, then any set of $k+1$ horses are white.

Let the proof of the induction step be as follows. Take a set of $k+1$ horses. Remove one horse; by induction hypothesis, all remaining horses are white. Now, put that horse back in and remove another horse. The remaining horses are again white by induction hypothesis, and so is the horse we took out the first time. Therefore, these $k+1$ horses are white, and as it was an arbitrary set of $k+1$ horses, induction step holds.

Therefore, all horses are white.

What is the trick here? The problem is that the induction step relies on some assumption about k that is not quite valid. More precisely, it needs a different base case. The problem occurs when the proof says "Now, put that horse back in and remove another horse". But then we need to guarantee that there is "another horse" in the set. It would be true if $k \geq 1$. However, for our base case we chose $k \geq 0$. Thus, the proof does not go through without the base case of $k = 1$ (which, as our common sense tells us, is not true: there are some horses out there that are not white).

This is just an example of caveats to watch out for when doing induction proofs: make sure there are no assumptions about k that could not be handled by the base case.

6.2 Variants of induction

You have seen already the Well-Ordering Principle, which can be considered an (equivalent) variant of induction. In this lecture we will look at another (also equivalent, although looking more powerful) variant of induction, called *strong* (or sometimes *complete*) induction. Here, instead of assuming that $P(i)$ holds for just one preceding element i, we assume that it holds for all elements from the base case up to (but not including) k, and then proceed with this stronger assumption to proving $P(k)$. We will prove the equivalence of the three principles later.

Definition 1 (Strong induction)**.** *Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:*

*1) **Base case**: for some $b \geq a$, $\forall a \leq c \leq b, P(c)$ is true.*

*2) **Induction step:** $(\forall i, b \leq i < kP(i)) \to P(k)$*

Then the statement

$$\textit{for all integers } n \geq a, \ P(n)$$

is true.

Example 1. Here is another way of solving the 3c and 5c coins problem, this time using strong induction. Recall that the goal is to prove that $\forall n \geq 8, \exists i, j \geq 0 \; n = 3i + 5j$.
Proof: Let $P(n)$ be $\exists i, j \geq 0 \; n = 3i + 5j$, as before.
`Base case:` This time, there are three base cases, $n = 8 = 3 \cdot 1 + 5 \cdot 1$, $n = 9 = 3 \cdot 3 + 5 \cdot 0$, and $n = 10 = 3 \cdot 0 + 5 \cdot 2$.
`Induction hypothesis` Assume that $\forall m, 8 \leq m < k$, $\exists i, j \geq 0 m = 3i + 5j$.
`Induction step.` As in the proof with well-ordering, consider $k - 3$. If $k - 3 \geq 8$, then there are i, j such that $k - 3 = 3i + 5j$ and so $k = 3(i + 1) + 5j$. Otherwise, k must be one of the three base cases $8, 9$ or 10, for which we know the corresponding i and j.

In this example, we made use of two things: first, strong induction allowed us to talk about the value of $k - 3$ as opposed to just $k - 1$. Second, we explicitly used base cases.