CS 2742 (Logic in Computer Science) – Winter 2014
Lecture 16

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5 Functions

Recall that a relation on n variables $R(x_1, \ldots, x_n)$ is a subset of the Cartesian product of domains of $x_1, \ldots, x_n$. A *function* is a special kind of relation that has exactly value of x_n for any tuple of values of $x_1 \ldots x_{n-1}$. Usually we write $f(x_1 \ldots x_{n-1}) = x_n$ to mean that R is a function and $R(x_1, \ldots, x_{n-1}, x_n)$ holds.

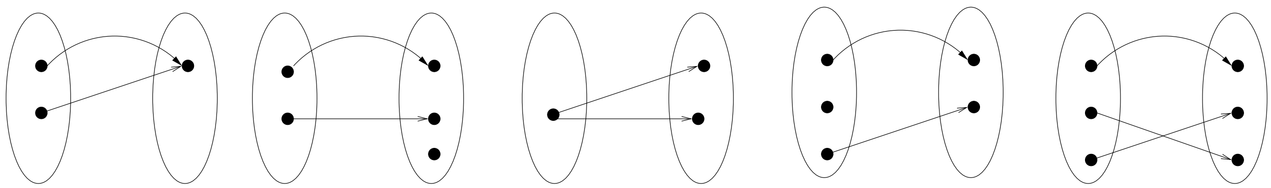
So just as we defined numbers using sets, we now defined functins and relations on numbers (and not just numbers: the variables can be anything).

Example 1. $f(x) = Mother(x)$ is a function, so is $f(x) = x^2$, so is $f(x) = x/y$.

Definition 1. *We often write functions as $f : X \to Y$ (read as "function f from X to Y") meaning that the tuples of variables of f come from X, and that the output value of f comes from Y. We call X the* domain *of f, and $\{y | x \in X \wedge f(x) = y\}$ a* range *of f, or* image *of X under f. A set Y is called* codomain*; the range of f is a subset of the codomain.*

Domain and codomain can be different sets: e.g., function counting the number of 0's in a binary string $f : \{0, 1\}^* \to \mathbb{N}$.

- Identity function: $f(x) = x$. Can be defined for any domain=codomain.
- Constant function: $f(x) = a$, where a does not change when x does. For example, $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 0$.
- Arithmetic functions: logarithmic function $f(x, y) = \log_x y$, exponential $f(x, y) = x^y$, addition, multiplication, division, subtraction, etc.



- Boolean functions: a function from strings of 0s and 1s of length n (denoted $\{0,1\}^n$) to $\{0,1\}$.

A function is defined by a formula if there is a formula which is true exactly on tuples of inputs + output of the function. E.g., a function $F : \mathbb{N} \to \mathbb{N}$ $f(x) = x + 1$ can be defined by $y > x \wedge \forall z\ (z \leq x \vee z \geq y)$. Sometimes a function is not well defined on a certain domain: e.g., $\sqrt{x}$ is not well-defined when both the domain and the range are natural numbers.

Definition 2. *Let $f : X \to Y$ be a function. Then f is* one-to-one *(or* injective*) iff $\forall x, y \in X\ (f(x) = f(y) \to x = y)$. A function is* onto *(or* surjective*) if $\forall y \in Y \exists x \in X (f(x) = y)$. A function is* bijective *if it is both one-to-one and onto.*

To prove that two sets are the same size, give a bijection (or give two functions, one a surjection and one an injection).

To prove that a function is one-to-one show that $f(x) = f(y) \to x = y$.

Example 2. For example, $f(x) = 4x + 1$, $f(x) = f(y)$ so 4x+1=4y+1 so $x = y$. On the other hand, $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^2$ is not one-to-one: as a counterexample take $x = -1$ and $y = 1$. Then $x \neq y$, but $x^2 = y^2$.

To prove that a function is onto, show that every element has a *preeimage*. To prove that it is not onto, show that there is an element in the codomain such that nothing maps into it.

Example 3. Consider again $f(x) = 4x+1$ over real numbers. There it is onto. Now consider it over integers. It is not onto integers.