# CS 2742 (Logic in Computer Science) – Winter 2014
# Lecture 14

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## 4.1 Negating quantified formulas

Recall again that to prove that something is not true "everywhere" we need to give a counterexample. Here we will do a few examples of negating first-order formulas with quantifiers.

**Example 1.** Here is an example of negating a formula with multiple quantifiers. We change all quantifiers to the opposite, and then negate the formula under the quantifiers as we would a propositional formula.

$$\begin{aligned}
&\neg(\exists x \forall y \forall z \exists u (\neg P(x,y) \vee (Q(z,u) \wedge z \neq y))\\
\iff &\forall x \exists y \exists z \forall u \neg(\neg P(x,y) \vee (Q(z,u) \wedge z \neq y))\\
\iff &\forall x \exists y \exists z \forall u (P(x,y) \wedge (\neg Q(z,u) \vee z = y))
\end{aligned}$$

**Example 2.** Consider the formula $\forall x\ (x^2 > x \vee x < 1)$. Suppose we want to prove that this formula is not true when the domain is real numbers $\mathcal{R}$. For that, we need to give a counterexample to the formula, that is, a real number such that $x^2 \not> x$ and $x \not< 1$. A counterexample that works here is $x = 1$, since $1^2 = 1$, not $> 1$, and $1 < 1$ does not hold either. The way we write it is
$\neg(\forall x\ (x^2 > x \vee x < 1) \iff \exists x\ \neg(x^2 > x \vee x < 1) \iff \exists x\ (x^2 \leq x \wedge x \geq 1)$ Here, we took a simplification one step further than usual, and wrote $\neg(x^2 > x)$ as $x^2 \leq x$, and the same for $x \geq 1$.

**Definition 1.** *An* instantiation *of a variable is a specific value that this variable is set to.*

For example, in the formula $x^2 > x \vee x < 1$ above we instantiated $x$ to be 1.

Now we can define what it means for one predicate formula to imply another, and for two formulas to be equivalent. When we say that $A(x,\ldots,z) \to B(x,\ldots,z)$ what we mean is that for every instantiation (sometimes called "interpretation" in this context) of free variables, if $A(x, \dots, z)$ is true on that instantiation then so is $B(x, \dots, z)$. Similarly, we say that $A$ is equivalent to $B$ (that is, $A(x, \dots, z) \iff B(x, \dots, z)$) if for every instantiation of free variables $A \iff B$) for that instantiation.

## 4.2 Derivations in predicate logic

One of the main tools in proving mathematical statements, and in deductive reasoning in general, is the rule of universal instantiation.

**Definition 2.** *The rule of* universal instantiation*: if some property is true of everything in a domain, then it is true of any particular thing in the domain.*

So if $\forall x\ x^2 \geq x$, then $5^2 > 5$. Another example says that every number is either even or odd ($\forall x\ (2|x \vee 2|x+1)$). Therefore, if we take some number $k$, then $k$ is either even or it is odd.

A classical example of reasoning using the rule of universal instantiation is the following:

> All men are mortal
> Socrates is a man
> $\therefore$ Socrates is mortal.

There are several ways to write this argument in predicate logic. The first will make use of the rule of universal instantiation under the assumption that the domain of the quantifier is "men". The second one that explicitly specifies the domain by using an implication, will do the rule of universal instantiation followed by modus ponens. Finally the third one, most closely resembling the original argument, will combine the universal instantiation and modus ponens into one rule, called *universal modus ponens*.

Let us consider predicates $Man(x)$ and $Mortal(x)$, which are true, respectively, on $x$ that are men, and $x$ that are mortal. Let $Men$ be the set of all men (this is the domain of the $\forall$ quantifier in the first example).

> $\forall x \in Men\ Mortal(x)$
> $\therefore Mortal(Socrates)$

> $\forall x(Man(x) \to Mortal(x)$
> $Man(Socrates) \to Mortal(Socrates)$
> $Man(Socrates)$
> $\therefore Mortal(Socrates)$

$$\forall x(Man(x) \rightarrow Mortal(x))$$
$$Man(Socrates)$$
$$\therefore Mortal(Socrates)$$

The general rule for the universal modus ponens, the rule used in the original form of the argument and the third translation into logic, is as follows

$$\forall x \ P(x) \rightarrow Q(x)$$
$P(a)$ for a particular $a$
$$\therefore Q(a).$$

Let us look at a more realistic mathematical proof using universal instantiation. Suppose in a piece of a proof goes as follows:

For all $x, m, n$, $x^m x^n = x^{m+n}$.
For all $x$, $x^1 = x$.
Therefore, $r^{k+1} r = r^{k+1} r^1 = r^{k+2}$.

Here, we instantiated $x = r, m = k+1, n = 1$. In the first equality in the last line, we used the second premise and in the second equality the first premise. We also used the fact that $1 + 1 = 2$.

**Puzzle 1.** A man walks into a bar and says to the barman: "pour everybody a drink! when I drink, everyone drinks!". After he finishes the round, he says again: "pour everybody a drink! when I drink, everyone drinks!". The crowd is quite pleased, until he says: "Give me the bill, I'll pay. When I pay, everybody pays!".

What does it have to do with logic, you may ask? Tell me, is there a man such that when he drinks, everybody drinks?