CS 2742 (Logic in Computer Science) – Winter 2014
Lecture 12

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February 12, 2014

5.4 Building new sets: power set, Cartesian product, relations.

A *power set* of a set A, denoted 2^A, is a set of all subsets of A. For example, if $A = \{1, 2, 3\}$ then $2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.

Let $|A|$ denote the number of elements of A (also called *cardinality*, especially when talking about infinite sets.) The size of the power set, as notation suggests, is $2^{|A|}$.

Theorem 1. *Let A be a finite set. Then the cardinality of 2^A is $2^{|A|}$.*

Proof. Suppose A has n elements. Now, every subset S of A can be represented by a binary string of length n, which would have a 1 in the positions corresponding to an element in S, and a 0 in places corresponding to elements not in S. For example, if $A = \{1, 2, 3\}$ as above, then $S\{1, 3\}$ is represented by a string 101, and $\emptyset$ is represented by a string 000. Now, the number of binary strings of length n is 2^n. Therefore, the number of possible subsets of A (and thus the elements of 2^A) is also 2^n. □

What if A is infinite? Still the size of the powerset (called *cardinality* in this context) will be larger. In one of the upcoming lectures we will talk about a technique called Diagonalization, due to Cantor, that can be used to show this.

Another useful notation is the Cartesian product, which will allow us to talk about ordered tuples of elements (pairs, triples, etc). A *Cartesian product* of sets $A_1 \dots A_n$, denoted $A_1 \times \cdots \times A_n$ is a set of ordered tuples $< a_1, a_2, ...a_n >$ such that $a_1 \in A_1 \wedge a_2 \in A_2 \wedge \cdots \wedge a_n \in A_n$. Note that an *ordered tuple* (a, b) is not the same as a set $\{a, b\}$: here the order of elements matters, so the tuple $< 1, 2 >$ is not the same as the tuple $< 2, 1 >$. For two sets, their Cartesian product is $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

For example, a Cartesian product of sets $\{3,4\}$ and $\{1,2,3\}$ is the set of pairs $\{(3,1),(3,2),$ $(3,3),(4,1),$ $(4,2),(4,3)\}$. Note that the pair $(4,3)$ is in the set, but the pair $(3,4)$ is not, because 4 is not an element of $\{1,2,3\}$.

Definition 1. *A relation on n variables $R(x_1,\ldots,x_n)$ is a subset of the Cartesian product of domains of $x_1,\ldots,x_n$.*

You often hear an expression "relational databases". Indeed, a standard way to describe a database is as a set of relations, where each parameter corresponds to a field in a database, and each tuple (element) to an item in the database. For example, a student database could have a relation $StudentInfo(name, number, address)$ which could be a subset of $Strings \times \mathbb{N} \times Strings$. That is, every item in this relation will consist of three elements, denoting the name, student number and address of a specific student. A database usually contains several different relations.

In the next topic, we will see how propositional logic can be extended to deal with relations, and to ask logical queries about them.

6 Predicate logic

Sometimes we encounter sentences that only have a truth value depending on some parameter. For example, $Even(x)$ which states that the number x is even can be true or false depending on the actual value of x. That is, $Even(5)$ is false, and $Even(10)$ is true.

It is convenient to think of predicates as propositions with parameters. Here, parameters can be numbers, items, etc and there can be infinitely many possibilities for a parameter value. For example, $x^2 > x$ is a predicate with an argument x, where we think of x as a number. Another predicate $Parent(x, y)$ could state that x is a parent of y. Here, it makes sense to think of x and y as people, or at least living creatures. Truth values of a predicate are defined for a given assignment of variables. For example, if $x = 2$, then $x^2 > x$ is true, and if $x = 0.5$, then $x^2 > x$ is false. We call a set of possible objects from which the values of a predicate can come from a *domain* of a predicate.

So what is the relation between a predicate, for example $Parent(x, y)$, and a relation $Parent$? A predicate is true iff the corresponding tuple of values is in the relation. For example, $Parent(John, Mary)$ is true if John is a parent of Mary, and the pair (John, Mary) is in the relation $Parent$. Usually we will use the notation $P(x, y, z)$ to mean a predicate, and just P to denote a set (relation); however sometimes I will abuse the notation and mix up these two concepts (especially when talking about databases).

6.1 Quantifiers

Without fixing the values of arguments of a predicate it is not possible to say if the predicate is true or false. That is, unless we want to say that the predicate is false for all possible values of its arguments (in the domain of this predicate). Here, we need to pay careful attention to what we mean by all possible values: $x^2 \geq x$ is true for and it is false for some rational and real numbers such as 0.5.

Quantifiers are the notational device that allows us to talk about all possible values of arguments and make sentences with truth values out of predicates.

Definition 2. *A formula $\forall x\ A(x)$, where $A(x)$ is a formula containing predicates, is true (on the domain of predicates) if is is true on every value of x from the domain. Here, $\forall$ is called a* universal quantifier*, usually pronounced as "for all ...".*

For example, $\forall x\ x^2 \geq x$ states that for every element from the domain the square of that element is greater than the element itself. This formula now has a truth value, provided we know the domain from which x comes from. If the domain is $\mathcal{Z}$, then the formula is true, and if the domain is $\mathcal{Q}$, then it is false. Often the domain is written explicitly: $\forall x \in \mathcal{Z}\ x^2 \geq x$, which is a shortcut for $\forall x\ (x \in \mathcal{Z} \to x^2 \geq x)$.

When we want to say that something is not true everywhere, all we need to do is to give a counterexample. E.g., to show that for $\mathcal{Q}$ it is not true that $\forall x\ x^2 \geq x$ it is enough to give one value on which $x^2 \geq x$ does not hold such as $x = 0.5$. We denote this with the second type of quantifiers, an *existential* quantifier.

Definition 3. *A formula $\exists x\ A(x)$, where $A(x)$ is a formula containing predicates, is true (on the domain of predicates) if is is true on* some *value of x from the domain. Here, $\exists$ is called a* existential quantifier*, usually pronounced as "exists ...".*

When doing boolean operations on formulas containing quantifiers, always remember that universal and existential quantifiers are opposites of each other. So,

$$\neg(\forall x\ A(x)) \iff \exists x\ \neg A(x) \qquad\qquad \neg(\exists x\ A(x)) \iff \forall x\ \neg A(x)$$

Now that we have this notation we can define what kinds of formulas we can construct using this language, the *first-order formulas.*

Definition 4. *A predicate is a first-order formula (possibly with free variables). A $\wedge, \vee, \neg$ of a first-order formula is a first-order formula. If a formula $A(x)$ has a free variable (that is, a variable x that occurs in some predicates but does not occur under quantifiers such as $\forall x$ or $\exists x$), then $\forall x\ A(x)$ and $\exists x\ A(x)$ are also first-order formulas.*

Note that this definition is very similar to the definition of propositional formulas except here there are predicates instead of propositions and there are quantifiers.

6.2 English and quantifiers

In English, the closest word to the universal quantifier is “all” or “every”. The closest word to the existential quantifier is “some” and “exists”. But there is one word that can be used as either a universal or an existential quantifier.That is the word *any*. Often we take it to mean a universal quantifier, as in “take any number greater than 1...” (that is, every number greater than 1 would work). But compare the following two sentences:

> “I will be happy if I do well in *every* class”.
> “I will be happy if I do well in *any* class”.

Here, the word “any” takes the meaning of an existential quantifier: that is, I’ll be happy if there exists some class in which I do well. Please keep this in mind when doing the translations.

Puzzle 8. The first formulation of the famous liar’s paradox, done by a Cretan philosopher Epimenides, stated “All Cretans are liars”. Is this a paradox?