CS 2742 (Logic in Computer Science) – Winter 2014
Lecture 11

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Before continuing, let us look at a simple example of sets:

Example 1 (Intervals on a real line)**.** Let $(-1, 0]$ and $[0, 1)$ be two intervals on a real line. $(-1, 0] \cup [0, 1) = (-1, 1)$, $(-1, 0] \cap [0, 1) = \{0\}$, $(-1, 0] - [0, 1) = (-1, 0)$.

Definition 1. *Two sets are disjoint if they have no common elements. Sets $A_1 \dots A_n$ form a partition of a set A if sets are pairwise disjoint (that is, $\forall i, j$ $A_i \cap A_j = \emptyset$ and their union forms the whole set A.*

Note that the rule of inclusion/exclusion simplifies greatly when sets are disjoint: in this case, the sum of the sizes of the disjoint sets is exactly the size of their union. This is used in probability theory.

Some subset relations:

1) $A \cap B \subseteq A$

2) $A \subseteq A \cup B$

3) Transitivity: if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

To prove that $A \subseteq B$ show that any element $x \in A$ is also $\in B$. Proof style: "suppose x is in A. Now show that $x \in B$." Now use logic of the definitions.

Now can prove properties of sets such as DeMorgan, by using "suppose x is in the left side... show that x is in the right side".

To prove something does not hold, find a counterexample.

Example 2. Show that it is not true that for all A, B, C, $(A - B) \cup (B - C) = A - C$. To prove this, find a counterexample, that is, sets A, B, C for which this does not hold. Let $A = \{1, 2\}$, $B = \{2\}$ and $C = \{1\}$. Then $A - C = \{2\}$, $A - B = \{1\}$, $B - C = \{2\}$ and the union is $\{1, 2\}$. Alternatively, think of an element in the LHS that is not in $A - C$: in this case, such an element is some element not in A.

5.1 Boolean algebras

As you have noticed, the algebra of sets is very similar to the algebra of propositions. This is because they are all examples of boolean algebras.

Definition 2. *A Boolean algebra is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following properties hold:*

- *Commutative laws: $a + b = b + a$ and $a \cdot b = b \cdot a$.*
- *Associative laws: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.*
- *Distributive laws: $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + c = (a + c) \cdot (b + c)$ (recall that the second one does not hold for the normal arithmetic $+$ and $\cdot$).*
- *Identity laws: $a + 0 = a$ and $a \cdot 1 = a$*
- *Complement laws: for each a there exists an element called negation of a and denoted $\bar{a}$ such that $a + \bar{a} = 1$, $a \cdot \bar{a} = 0$.*

In the case of propositional logic, 0 is F, 1 is T and there are no other elements, so it is sufficient to say that $\bar{T} = F$ and $\bar{F} = T$ (in that setting, $\neg$ is used for complementation). In set theory, 0 and 1 are $\emptyset$ and the universe U, respectively, and negation of every set is its complement. Now, properties of Boolean algebras such as DeMorgan's law can be derived from these axioms.

Example 3 (Idempotent identity)**.** Show that $a + a = a$.

$$\begin{aligned}
a &= a + 0 && \text{because 0 is the identity for } + \\
&= a + (a \cdot \bar{a}) && \text{by the complement law for } \cdot \\
&= (a + a) \cdot (a + \bar{a}) && \text{by the distributive law} \\
&= (a + a) \cdot 1 && \text{by the complement law for } + \\
&= a + a && \text{because 1 is the identity for } +
\end{aligned}$$

Now, we obtain that the Idempotent identity holds for propositional logic and for set theory, since both are examples of Boolean algebras.