MACM 101, D2, 10/01/2007. Lecture 2.

Puzzle of the day:

How many ways can you permute the letters of the word "Osoyoos" so that each sequence is different (ignoring case)? E.g., "oooossy" can be one sequence and "sooosoy" a different one.

How many such sequences do not have the two "s" together?

(Osoyoos is a small town in BC near US border).

Solution: If all the letters were distinct, there would be 7! different sequences. However, there are 4 indistinguishable o's and 2 identical "s", so we ignore the order of o's (4!) and s's (2!). Therefore, the total number of distinct sequences is 7!/(4!2!) = 105.

Now consider the case when "s" are not together. Solution 1: count the number of sequences with "ss", and substract from the total. There are 6!/4!=30 sequences with "ss" (treating "ss" as one letter). So the number of sequences with "s" separate is 105-30=75.

Solution 2 (more general). Consider all sequences of 4 o and 1 y. There are 5 of these (5!/4!). Look at such a sequence `␣o␣o␣o␣o␣y␣` There are 6 possible places between the letters where s can go, for example, in "`␣oso␣o␣o␣ys`" s's take second and 6th place. By the formula for combinations, there are (6 choose 2) ways of placing 2 identical letters in 2 of 6 identical spots. The final answer, by the rule of product, is the product of permutations of 4 o and 1 y, and $\binom{6}{2}$. This gives the answer (5!/4!)*(6!/2!4!)=5*15=75.

Topic: binomial theorem

Attributed to Blaise Pascal (XVII AD) by europeans, but mentioned by Yang Hui (China, XIII AD), Omar Khayyam (Persia, XI AD) and even Pingala (India, III BC).

Theorem 1 (binomial theorem).

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \ldots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0$$

Example 1. *In $(x+y)^4$, the coefficient at x^2y^2 is $\binom{4}{2} = 6$. Because there are 4 x's and 4 y, one from each sum, and each monomial has exactly one letter (x or y) from each of the four $(x+y)$. That is, x^2y^2 can be $x_1x_2y_3y_4$ or $x_2x_4y_1y_3$ and so on. Note that once you've chosen indices of x's, ys are fixed. The number of ways to choose two x's out of 4 possibilities is $\binom{4}{2}$, which is the coefficient.*

The proof of the binomial theorem generalizes this argument. Suppose the monomial is x^ky^{n-k}. The coefficient will be the number of ways to choose k indices of x's out of n possibilities, which is $\binom{n}{k}$.

Pascal triangle (see textbook example 3.14, page 133). Top row is 1, second row is 1 1, third is 1 2 1 and so on, centered. Each row is binomial coefficients for some n starting with $n = 0$ for the first row (with single 1). Next row is computed from previous by summing up the two numbers above/left and above/right,with 1 at the edges. E.g., in the third row sits under 1 from the first row, between and under 1 and 1 from the second row, and is computed as the sum of 1's from the second row. In general, the formula is

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Intuition: number of ways to take k cards out of a deck of n is the same as setting aside one card, computing number of ways to take $k-1$ cards out of the remaining ones (and adding one we set aside to get k cards), plus all ways to take k cards out of the remainded of the deck without one we set aside.

Corollary 1. *(of binomial theorem)*
a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$
b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$

Proof of a). Look at the expansion of $(1+1)^n$ by binomial theorem, get an expression in a), but $(1+1)^n = 2^n$.

Proof of b). The expression in b) is the expansion by binomial theorem of $(1-1)^n$.