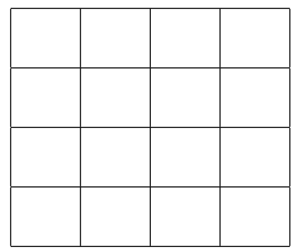
MACM 101, D2, 08/01/2007. Lecture 1.

Puzzle of the day:

Consider a square divided vertically and horizontally by 3 lines each



How many squares are there in this picture? What if the square is not 4x4 but 5x5? $n \times n$?

Solution: There are 4*4=16 squares of size 1x1,3*3=9 of size 2x2, 2*2=4 of size 3x3 and one (1*1=1) square 4x4. Adding these four numbers, get 16+9+4+1=30.

The case 5x5 is reducible to 4x4: it is the same number of squares larger by 1 (e.g., now there are 2*2=4 squares of size 4x4 rather than 3x3). Plus, there are 5*5=25 squares of size 1x1. This gives the total of 30+25=55 squares.

The general case becomes the sum of squares:

$1 * 1 + 2 * 2 + 3 * 3 + ...n * n = (n * (n + 1)(2n + 1))/6$

This formula can be proven by induction (later in the class).

1 Counting.

Back to the example with the squares. We used two rules, a rule of sum and a rule of product.

Rule of Sum: if A occurs m times, B occurs n times, then $A \vee B$ occurs n+m times (here, A and B are disjoint).

Rule of Product: if A occurs m times, B occurs n times, then $A \wedge B$ occurs n*m times.

Both need an assumption that A and B are independent (e.g., don't overlap).

We used the rule of *product* when multiplying number of columns by number of rows to get number of squares of a given size, e.g, 3*3=9 squares of size 2x2. We used the rule of *sum* when adding up number of squares of different sizes.

Combinations (review): How many ways to get three cards out of a deck of 52? If we don't care about the order and all cards are distinct, 52!/(3!49!).

Formula: $\binom{n}{k}$ ("n choose k") is $\frac{n!}{k!(n-k)!}$.

Number of *permutations* of cards 2,5, 10: three places, three cards. The first place can take any of three cards, second place any of the two remaining, third place takes the last card. Therefore, number of permutations of 3 different cards is 3*2*1=3!.

Formula: number of permutations of n objects is $n!$

Consider all permutations that have given three cards in front. If we choose by taking the first three cards, we don't care about 1) the order of these three cards 2) the order of the remaining 49 cards. So all sequences that have, say, 2 of hearts, 5 of spades, 10 of clubs in front will all "look the same" to us. The number of sequences of the first three cards is 3!. The number of ways to permute the remaining 49 cards is 49!. Therefore, the total number of ways to choose three cards out of a deck is 52!/(3!49!).

Deriving formula for $\binom{n}{k}$: consider number of all possible permutations of n objects (e.g., of 52 cards). It will be n!. Ignore the order of k objects we are choosing, we get $n!/k!$. Finally, ignore the order of the remaining objects $(n-k)!$, arriving to $n!/(k!(n-k)!)$

Note: when we don't want to ignore the order of objects we have chosen, we get *arrangements*. The number of arrangements is computed by formula $n!/(n-k)! = k! * \binom{n}{k}$.