

# CS 2742 (Logic in Computer Science)

## Lecture 6

Antonina Kolokolova

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### 2.1 Valid and invalid arguments

We will start with a (propositional form of) Aristotle's classic example of a valid argument (his original version had a quantifier; we will get to that in a few lectures).

If Socrates is a man, then Socrates is mortal  
Socrates is a man  
 $\therefore$  Socrates is mortal

Terminology: the final statement is called *conclusion*, the rest are *premises*. The symbol  $\therefore$  reads as *therefore*. An argument is *valid* if no matter what statements are substituted into variables, if all premises are true then the conclusion is true. That is, if there are formulas  $P_1, \dots, P_k$  that are all the premises (one per line, as above), and a formula  $C$  is the conclusion, then an argument is valid iff  $P_1 \wedge P_2 \wedge \dots \wedge P_k \rightarrow C$  is a tautology.

The valid form of argument is called *rules of inference*. The most known is called *Modus Ponens* ("method of affirming"). Its contrapositive is called *Modus Tollens* ("method of denying"):

*Modus Ponens*

If  $p$  then  $q$   
 $p$   
 $\therefore q$

*Modus Tollens*

If  $p$  then  $q$   
 $\neg q$   
 $\therefore \neg p$

This is another way to describe "proof by contrapositive". Similarly we can write the proof by cases, by contradiction, by transitivity and so on. They can be derived from the original logic identities. For example, modus ponens becomes  $((p \rightarrow q) \wedge p) \rightarrow q$ .

**Example 1.** The general form of the proof by cases above can be written as follows:

$p \vee q$	“ $n$ is even or $n$ is odd”
$p \rightarrow r$	“if $n$ is even then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ “
$q \rightarrow r$	“if $n$ is odd then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ “
$\therefore r$	“therefore $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ “

To show that an argument is invalid give a truth table where all premises are true and the conclusion is false. That is, show that a propositional formula of the form  $\wedge(\text{of premises}) \rightarrow \text{conclusion}$  is not a tautology.

**Example 2.** For example, although modus ponens is valid, the following is not a valid inference:

$p \rightarrow q$	
$q$	INVALID ARGUMENT!
$\therefore p$	

We can write it as  $((p \rightarrow q) \wedge q) \rightarrow p$  and show that this is not a tautology using a truth table.

$p$	$q$	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$(p \rightarrow q) \wedge q \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	<b>F</b>
F	F	T	F	T

The truth table showed us a situation when both premises  $(p \rightarrow q)$  and  $q$  are true, but the conclusion  $p$  is false. Therefore,  $((p \rightarrow q) \wedge q) \rightarrow p$  is not a tautology and thus the argument based on it is not a valid argument.

However, note that if any of the premises are false, a valid argument can produce a most weird conclusion: remember that if  $p$  is false in  $p \rightarrow q$ , then  $p \rightarrow q$  is true for any value of  $q$ . Thus, if a premise is false, a false conclusion can be reached (even though an argument is valid); a true conclusion can be reached as well.

A brilliant example that shows that falsity can imply anything via a valid argument was presented by the famous logician Bertrand Russell (author of the “Russell’s paradox” that we will see later in the course.)

**Example 3.** Bertrand Russell: “If  $2+2=5$ , then I am the Pope”.<sup>1</sup>

*Proof.* If  $2+2=5$  then  $1=2$  by subtracting 3 from both sides.

Bertrand Russell and Pope are two people.

Since  $1=2$ , Bertrand Russell and Pope are one person.

□

Note that the steps of this proof are perfectly fine logically. Every line follows from the previous line correctly. The strange conclusion exclusively from  $2+2=5$  being a false statement.

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<sup>1</sup>According to other sources, the statement Bertrand Russell was proving was “if  $1+1=1$ , then I am the Pope”. In this case, the first line can be omitted.