

CS 2742 (Logic in Computer Science)

Lecture 3

Antonina Kolokolova

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1.1 Logical equivalences

Recall the puzzle from the previous class: on some island, there are knights (who always tell the truth) and knaves (who always lie). You meet two islanders (call them A and B) and hear the first one say “at least one of us is a knave”. Can you tell whether the islanders are knights or knaves and which islander is which?

We solve this puzzle using a truth table. Take p :”A is a knight” and q : “B is a knight.”. Then the sentence “At least one of us is a knave” is translated as $(\neg p \vee \neg q)$, since being a knave is the negation of being a knight. Now, we want to know when the truth value of this sentence $(\neg p \vee \neg q)$ is the same of the truth value of p : that is, if A is a knight, then what he said must be true, and if A is a knave, then what he said must be false. This we can state as A is a knight if and only if “at least one of us is a knave” is true. We represent this as a truth table as follows:

p	q	$(\neg p \vee \neg q)$	$p \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F
T	F	T	T
F	T	T	F
F	F	T	F

As you can see, the only scenario when what A says corresponds correctly with A’s being a knight/knave is the second line: that is, when A is a knight and B is a knave. Let us introduce the notation \iff to mean *logical equivalence* (that is, two formulas having the same truth values for any truth assignment to their variables). A better way way of stating the last condition is as a logical equivalence of p and $f(\neg p \vee \neg q)$, that is, $p \iff (\neg p \vee \neg q)$.

Definition 1 We say that two formulas are logically equivalent ($A \iff B$) if they have the same truth value under any truth assignment.

That is, in a truth table the columns of logically equivalent formulas are identical.

Theorem 1 *Two formulas A and B are logically equivalent if and only if a formula $A \leftrightarrow B$ is a tautology.*

For the proof, Check that the columns for A and B are the same if $A \leftrightarrow B$ is a tautology, and different if it is not. Although A and B are formulas here, this still can be checked with just a 4-line truth table.

A useful property of logically equivalent formulas, called *substitution*, is that if in any formula you replace a subformula by another that is logically equivalent to it, then the value of the whole formula would not change. For example, $p \wedge (q \vee \neg q)$ is logically equivalent to $(p \wedge T)$, which is in turn equivalent to p (can check this using a truth table). So if in a formula there is an occurrence of $p \wedge (q \vee \neg q)$ it can be safely replaced with just p , thus simplifying the formula. Another example that you see quite often is substituting $(p \rightarrow q)$ with $(\neg p \vee q)$, so, for example, $r \wedge (p \rightarrow q) \vee \neg p$ is logically equivalent to $r \wedge (\neg p \vee q) \vee \neg p$.

1.2 Logical identities

Now that we have a notion of logical equivalences we can talk about a few identities in propositional logic. We will list them as pairs of equivalent formulas.

Name	\wedge -version	\vee -version
Double negation	$\neg\neg p \iff p$	
DeMorgan's laws	$\neg(p \wedge q) \iff (\neg p \vee \neg q)$	$\neg(p \vee q) \iff (\neg p \wedge \neg q)$
Commutativity	$(p \wedge q) \iff (q \wedge p)$	$(p \vee q) \iff (q \vee p)$
Associativity	$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$	$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$
Distributivity	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$ $p \wedge F \iff F$	$p \vee F \iff p$ $p \vee T \iff T$
Idempotence	$p \wedge p \iff p$	$p \vee p \iff p$
Absorption	$p \wedge (p \vee q) \iff p$	$p \vee (p \wedge q) \iff p$

Notice again (as working our way to boolean algebras) that many of these identities behave just like algebraic and arithmetic identities, with \wedge behaving like \times , \vee like $+$, T like 1 and F like 0. For example, commutativity and associativity laws are the same as for numbers: $(3 + 2) + 5 = 3 + (2 + 5)$. One notable exception is that with numbers there is just one form of distributed law, namely the \wedge form ($a \times (b + c) = (a \times b) + (a \times c)$), and the \vee form does not hold ($a + b \times c \neq (a + b) + (a + c)$), whereas in logic both forms are true.

It is convenient to treat \rightarrow and \leftrightarrow as “syntactic sugar”, and define them to be their equivalent formulas with just \vee, \wedge, \neg . That is, we will say that $(p \rightarrow q) \iff (\neg p \vee q)$ by definition, and so is $(p \leftrightarrow q) \iff (\neg p \vee q) \wedge (\neg q \vee p)$.