CS2742 midterm test 1 study sheet

Propositional logic:

- *Propositional statement*: expression that has a truth value (true/false). It is a *tautology* if it is always true, *contradiction* if always false.

- *Logic connectives*: negation ("not") $\neg p$, conjunction ("and") $p \wedge q$, disjunction ("or") $p \vee q$, implication $p \to q$ (equivalent to $\neg p \vee q$), biconditional $p \leftrightarrow q$ (equivalent to $(p \to q) \wedge (q \to p)$). The order of precedence: $\neg$ strongest, $\wedge$ next, $\vee$ next, $\to$ and $\leftrightarrow$ the same, weakest.

- If $p \to q$ is an implication, then $\neg q \to \neg p$ is its *contrapositive*, $q \to p$ a *converse* and $\neg p \to \neg q$ an *inverse*. An implication is equivalent to its contrapositive, but not to converse/inverse or their negations. A negation of an implication $p \to q$ is $p \wedge \neg q$ (it is not an implication itself!)

- A *truth table* has a line for each possible values of propositional variables (2^k lines if there are k variables), and a column for each variable and subformula, up to the whole statement. The cells of the table contain T and F depending whether the (sub)formula is true for the corresponding values of variables.

- A *truth assignment* is a string of values of variables to the formula, usually a row with values of first several columns in the truth table (number of columns = number of variables). A truth assignment is *satisfying* the formula if the value of the formula on these variables is T, otherwise the truth assignment is *falsifying.* A truth assignment can be encoded by a formula that is a $\wedge$ of variables and their negations, with negated variables in places that have F in the assignment, and non-negated that have T. For example, $x = T, y = F, z = F$ is encoded as $(x \wedge \neg y \wedge \neg z)$.It is an encoding in a sense that this formula is true only on this truth assignment and nowhere else.

- Two formulas are *logically equivalent* if they have the same truth table. The most famous example of logically equivalent formulas is $\neg(p \vee q) \iff (\neg p \wedge \neg q)$ (with a dual version $\neg(p \wedge q) \iff (\neg p \vee \neg q)$) where p and q can be arbitrary (propositional, here) formulas. These pairs of logically equivalent formulas are called *DeMorgan's law.*

- There are several other important pairs of logically equivalent formulas, called *logical identities* or *logic laws.* We will talk more about them when we talk about Boolean algebras. Here, just remember that $F \wedge p \iff p \wedge \neg p \iff F$, $F \vee p \iff T \wedge p \iff p$ and $T \vee p \iff p \vee \neg p \iff T$.

- A set of logic connectives is called *complete* if it is possible to make a formula with any truth table out of these connectives. For example, $\neg, \wedge$ is a complete set of connectives, and so is the Sheffer's stroke $|$ (where $p|q \iff \neg(p \wedge q)$), also called NAND for "not-and". However, $\vee, \wedge$ is not a complete set of connectives because it is impossible to express a truth table with 0 when all variables are 1 with them.

- An *argument* consists of several formulas called *premises* and a final formula called a *conclusion.* If we call premises $A_1 \dots A_n$ and conclusion B, then an argument is *valid* iff premises imply the conclusion, that is, $A_1 \wedge \cdots \wedge A_n \to B$. We usually write them in the following format:

  Today is either Thursday or Friday
  On Thursdays I have to go to a lecture

Today is not Friday (alternatively, On Friday I have to go to the lecture)

$\therefore$ I have to go to a lecture today

- A valid form of argument is called *rule of inference.* The most prominent such rule is called *modus ponens.*

$$\begin{array}{l} p \to q \\ p \ \overline{\qquad\qquad} \\ \therefore q \end{array}$$

- There are several main types of proofs depending on the types of rules of inference used in the proof. The main ones are *proof by contrapositive, by contradiction* and *by cases.*

- There are two main normal forms for the propositional formulas. One is called *Conjunctive normal form* (CNF) and is an $\wedge$ of $\vee$ of either variables or their negations (here, by $\wedge$ and $\vee$ we mean several formulas with $\wedge$ between each pair, as in $(\neg x \vee y \vee z) \wedge (\neg u \vee y) \wedge x$. A *literal* is a variable or its negation (x or $\neg x$, for example). A $\vee$ of (possibly more than 2) literals is called a *clause*, for example $(\neg u \vee z \vee x)$, so a CNF is true for some truth assignment whenever this assignment makes each of the clauses is true, that is, each clause has a literal that evaluates to true under this assignment.. A *Disjunctive normal form* (DNF) is like CNF except the roles of $\wedge$ and $\vee$ are reversed. A $\wedge$ of literals in a DNF is called a *term.* To construct a DNF and a CNF, start from a truth table and then for every satisfying truth assignment $\vee$ its encoding to a DNF, and for every falsifying truth assignment $\wedge$ the negation of its encoding to the CNF, and apply DeMorgan's law. This may result in a very large CNFs and DNFs, comparable to the size of the truth table itself ($2^{\text{number of variables}}$). Alternatively, a CNF can be constructed from a formula by assigning a new variable v_i to every connective and rewriting the formula as a conjunction of v_1 and expressions defining $v_i's$, each containing just two other variables, and then converting these expressions into small CNFs using truth tables. For example, a formula $(x \vee y) \to (\neg z)$ can be converted to a CNF by introducing variables v_1 and v_2, then writing $v_1 \wedge (v_1 \leftrightarrow (v_2 \to \neg z)) \wedge (v_2 \leftrightarrow (x \vee y))$, then replacing each part by a CNF using truth tables.

- A *resolution proof system* is used to find a contradiction in a formula (and, similarly, to prove that a formula is a tautology by finding a contradiction in its negation). Resolution starts with a formula in a CNF form, and applies the rule "from clause $(C \vee x)$ and clause $(D \vee \neg x)$ derive clause $(C \vee D)$ until a falsity F (equivalently, empty clause ()) is reached (so in the last step one of the clauses being *resolved* contains just one variable and another clause being resolved contains just that variable's negation. Resolution can be used to check the validity of an argument by running it on the $\wedge$ of all premises (converted, each, to a CNF) $\wedge$ together with the negation of the conclusion.

- *Boolean functions* are functions which take as argument boolean (ie, propositional) variables and return 1 or 0 (or, the convention here is 1 instead of T, and 0 instead of F). Each Boolean function on n variables can be fully described by its truth table. A size of a truth table of a function on n variables is 2^n. Even though we often can have a smaller description of a function, vast majority of Boolean functions cannot be described by anything much smaller. Every Boolean function can be described by a CNF or DNF, using the above construction.

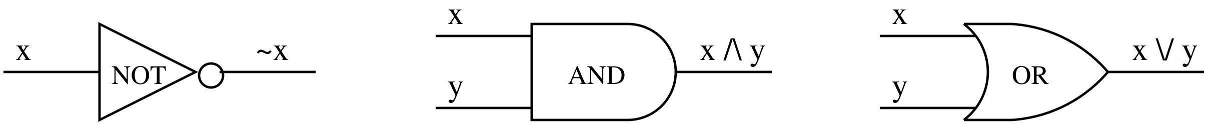

Figure 1: Types of gates in a digital circuit.

- *Boolean circuits* is a generalization of Boolean formulas in which we allow to reuse a part of a formula rather than writing it twice. To make a transition write Boolean formulas as trees and reuse parts that are repeating. The connectives become *circuit gates*.

  It is possible to have more than 2 inputs into an AND or OR circuit, but not a NOT circuit.

  It is possible to construct arithmetic circuits (e.g., for doing addition on numbers) by using a Boolean circuit to compute each bit of the answer separately.