CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 4

Antonina Kolokolova

September 13, 2011

1.1 More examples of simplifying formulas

Here we go over a few more examples similar to your assignment, simplifying formulas until they are as small as we can (reasonably easily) get using the rules from the last lecture.

Example 1.

$$
\begin{array}{llr}
 & p \leftrightarrow ((q \wedge \neg r) \rightarrow q) & \\
\iff & p \leftrightarrow (\neg(q \wedge \neg r) \vee q) & \text{Definition of } \rightarrow \\
\iff & p \leftrightarrow ((\neg q \vee \neg\neg r) \vee q) & \text{DeMorgan} \\
\iff & p \leftrightarrow ((\neg\neg r \vee \neg q) \vee q) & \text{Commutativity} \\
\iff & p \leftrightarrow (\neg\neg r \vee (\neg q \vee q)) & \text{Associativity (dropping parentheses)} \\
\iff & p \leftrightarrow (\neg\neg r \vee T) & \text{Definition of T} \\
\iff & p \leftrightarrow T & \text{Identity} \\
\iff & p & \text{Because } \leftrightarrow \text{ is an equivalence}
\end{array}
$$

The last step could be done more formally as follows:

$$
\begin{array}{llr}
 & p \leftrightarrow T & \\
\iff & (p \rightarrow T) \wedge (T \rightarrow p) & \text{Definition of } \leftrightarrow \\
\iff & (\neg p \vee T) \wedge (\neg T \vee p) & \text{Definition of } \rightarrow \\
\iff & (\neg p \vee T) \wedge (F \vee p) & \text{Definition of } F \\
\iff & T \wedge (F \vee p) & \text{Identity} \\
\iff & (F \vee p) & \text{Identity} \\
\iff & p & \text{Identity}
\end{array}
$$

1.2 Conditional statements

Conditional statements are ones of the form "if p then q", $p \to q$. Recall that we logically define $p \to q \iff (\neg p \vee q)$.

We use the following terminology when talking about conditional statements:

1) *Contrapositive* of $p \to q$ is $\neg q \to \neg p$. True whenever the original implication is.

   Proof. Recall that $(p \to q) \iff (\neg p \vee q)$. Now,

$$\begin{aligned} & \neg q \to \neg p \\ \iff & (\neg\neg q \vee \neg p) && \text{Definition of } \to \\ \iff & (q \vee \neg p) && \text{Double negation} \\ \iff & (\neg p \vee q) && \text{Commutativity} \iff (p \to q) \qquad \text{Definition of } \to \end{aligned}$$

   Thus, a contrapositive of an if-then statement is logically equivalent to the original statement. □

2) *Converse* and *inverse*: $q \to p$ and $\neg p \to \neg q$. Contrapositives of each other, can have a different truth value from $p \to q$.

3) *Sufficient* condition: p is sufficient for q if $p \to q$. The "if" part of if-then.

4) *Neccessary* condition: p is necessary for q if $\neg p \to \neg q$, that is, $q \to p$. The "then" part of "if-then".

5) If and only if (p iff q, $p \leftrightarrow q$) means $(p \to q) \wedge (q \to p)$.

Puzzle 3. What is the value of $2 + 2 = 4$?