CS 2742 (Logic in Computer Science) Lecture 3

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1.1 Logical equivalences

Recall the puzzle from the previous class: on some island, there are knights (who always tell the truth) and knaves (who always lie). You meet two islanders (call them A and B) and hear the first one say "at least one of us is a knave". Can you tell whether the islanders are knights or knaves and which islander is which?

We solve this puzzle using a truth table. Take p :"A is a knight" and q : "B is a knight.". Then the sentence "At least one of us is a knave" is translated as $(\neg p \vee \neg q)$, since being a knave is the negation of being a knight. Now, we want to know when the truth value of this sentence $(\neg p \vee \neg q)$ is the same of the truth value of p: that is, if A is a knight, then what he said must be true, and if A is a knave, then what he said must be false. This we can state as A is a knight if and only if "at least one of us is a knave" is true. We represent this as a truth table as follows:

p	q	$(\neg p \vee \neg q)$	$p \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F
T	F	T	T
F	T	T	F
F	F	T	F

As you can see, the only scenario when what A says corresponds correctly with A's being a knight/knave is the second line: that is, when A is a knight and B is a knave. Let us introduce the notation $\iff$ to mean *logical equivalence* (that is, two formulas having the same truth values for any truth assignment to their variables). A better way way of stating the last condition is as a logical equivalence of p and $f(\neg p \vee \neg q)$, that is, $p \iff (\neg p \vee \neg q)$.

Definition 1 *We say that two formulas are* logically equivalent *($A \iff B$) if they have the same truth value under any truth assignment.*

That is, in a truth table the columns of logically equivalent formulas are identical.

Theorem 1 *Two formulas A and B are logically equivalent if and only if a formula $A \leftrightarrow B$ is a tautology.*

For the proof, Check that the columns for A and B are the same if $A \leftrightarrow B$ is a tautology, and different if it is not. Although A and B are formulas here, this still can be checked with just a 4-line truth table.

A useful property of logically equivalent formulas, called *substitution*, is that if in any formula you replace a subformula by another that is logically equivalent to it, then the value of the whole formula would not change. For example, $p \wedge (q \vee \neg q)$ is logically equivalent to $(p \wedge T)$, which is in turn equivalent to p (can check this using a truth table). So if in a formula there is an occurrence of $p \wedge (q \vee \neg q)$ it can be safely replaced with just p, thus simplifying the formula. Another example that you see quite often is substituting $(p \rightarrow q)$ with $(\neg p \vee q)$, so, for example, $r \wedge (p \rightarrow q) \vee \neg p$ is logically equivalent to $r \wedge (\neg p \vee q) \vee \neg p$.

1.2 Logical identities

Now that we have a notion of logical equivalences we can talk about a few identities in propositional logic. We will list them as pairs of equivalent formulas.

Name	$\wedge$-version	$\vee$-version
Double negation	$\neg\neg p \iff p$	
DeMorgan's laws	$\neg(p \wedge q) \iff (\neg p \vee \neg q)$	$\neg(p \vee q) \iff (\neg p \wedge \neg q)$
Commutativity	$(p \wedge q) \iff (q \wedge p)$	$(p \vee q) \iff (q \vee p)$
Associativity	$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$	$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$
Distributivity	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
	$p \wedge F \iff F$	$p \vee T \iff T$
Idempotence	$p \wedge p \iff p$	$p \vee p \iff p$
Absorption	$p \wedge (p \vee q) \iff p$	$p \vee (p \wedge q) \iff p$

Notice again (as working our way to boolean algebras) that many of these identities behave just like algebraic and arithmetic identities, with $\wedge$ behaving like $\times$, $\vee$ like $+$, T like 1 and F like 0. For example, commutativity and associativity laws are the same as for numbers: $(3+2)+5 = 3+(2+5)$. One notable exception is that with numbers there is just one form of distributed law, namely the $\wedge$ form $(a \times (b+c) = (a \times b) + (a \times c)$, and the $\vee$ form does not hold $(a + b \times c) \neq (a \times b) + (a \times c))$, whereas in logic both forms are true.

It is convenient to treat $\rightarrow$ and $\leftrightarrow$ as "syntactic sugar", and define them to be their equivalent formulas with just $\vee, \wedge, \neg$. That is, we will say that $(p \rightarrow q) \iff (\neg p \vee q)$ by definition, and so is $(p \leftrightarrow q) \iff (\neg p \vee q) \wedge (\neg q \vee p)$.

1.3 Simplifying propositional formulas.

Now we can apply these identities to simplify propositional formulas.

Example 1

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\begin{aligned}
& (p \wedge q) \vee \neg(\neg p \vee \neg q) & \\
\iff & (p \wedge q) \vee (\neg\neg p \wedge \neg\neg q) & \text{Apply DeMorgan's} \\
\iff & (p \wedge q) \vee (p \wedge q) & \text{Double Negation (twice)} \\
\iff & (p \wedge q) & \text{Idempotence}
\end{aligned}
$$

Notice that the logic identities are stated only for the logical connectives $\wedge, \vee, \neg$. In order to deal with $\rightarrow$ and $\iff$ we use their definitions: for example, $A \rightarrow B$ becomes $\neg A \vee B$.