CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 17

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Definition 1. *A Boolean algebra is a set B together with two operations, generally denoted $+$ and $\cdot$, such taht for all a and b in B both $a+b$ and $a \cdot b$ are in B and the following properties hold:*

- *Commutative laws: $a+b=b+a$ and $a \cdot b = b \cdot a$.*
- *Associative laws: $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.*
- *Distributive laws: $(a+b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + c = (a+b) \cdot (a+c)$ (recall that the second one does not hold for the normal arithmetic $+$ and $\cdot$).*
- *Identity laws: $a+0=a$ and $a \cdot 1 = a$*
- *Complement laws: for each a there exists an element called negation of a and denoted $\bar{a}$ such that $a+\bar{a}=1$, $a \cdot \bar{a} = 0$.*

Proofs in Boolean algebra:

Example 1 (Idempotent identity)**.** Show that $a+a=a$.

$$\begin{aligned}
a &= a+0 && \text{because 0 is the identity for } + \\
&= a+(a \cdot \bar{a}) && \text{by the complement law for } \cdot \\
&= (a+a) \cdot (a+\bar{a}) && \text{by the distributive law} \\
&= (a+a) \cdot 1 && \text{by the complement law for } + \\
&= a+a && \text{because 1 is the identity for } +
\end{aligned}$$

6 Soundness and completeness of Boolean algebra

Boolean algebra has a very useful property: it is both sound and complete. We will see later in the course that any sufficiently powerful theory (such as ZFC) cannot be both; however, Boolean algebra is just weak enough to avoid this restriction.

Recall that soundness means that everything derivable from the axioms using the rules of inference is valid (true), and every valid formula is derivable from the axioms. An example of a theory that is not sound would be a theory which proves a formula and its negation, e.g, $0 = 1$ and $0 \neq 1$. Since everything follows from False, and $\phi \wedge \neg\phi$ is false, such a theory would imply everything.

To see that Boolean algebra is sound, just notice that the axioms are true facts, and that the derivation rules are of the form if $x = y$ then x can be replaced with y in the formula, and also simple rules of the form $x = x$, if $x = y$ then $y = x$, and if both $x = y$ and $y = z$ then $x = z$ (here, x, y, z can mean the whole formulas, more than just the variables). These derivation rules preserve the soundness. For the actual proof we need some concepts we will be developing in a few lectures.

Completeness is a more interesting property. To show that every valid statement in the language of Boolean algebra is derivable from the axioms, we will use truth tables. In particular, we will show that every formula can be converted, using the axioms and inference rules of Boolean algebra, to a canonical DNF (sometimes called normal form, or a complete DNF). This is a DNF obtained from the truth table by taking an OR of conjunctions (ANDs) of all literals for every line in the truth table that is true, with a literal being the variable if the value of that variable was true, and its negation if it was false. Those are exactly the DNFs we were constructing earlier in this course.

Now, to see that every formula can be converted to such a DNF, think of the following steps. First, use DeMorgan to push all negations onto variables, canceling out double negations when needed (e.g., $\overline{x + \bar{y}}$ becomes $\bar{x} \cdot y$). Then, use the distributive axiom to make it into a DNF. Then, for every term missing some variables make copies of this term for every possible combinations of values of missing variables (e.g., $(\bar{x} \cdot y)$ in a formula which also has z and no other variables becomes $(\bar{x} \cdot y \cdot z) + (\bar{x} \cdot y \cdot \bar{z})$. Finally, rearrange the terms so that the variables come in an order as they would in a truth table (e.g., alphabetical order). Now, if two formulas are logically equivalent (that is, have the same truth table) they would be converted to syntactically identical formulas, and from the definition of equality, any formula is equal to itself.