CS 2742 (Logic in Computer Science) – Fall 2011
Lecture 15

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Before continuing, let us look at a simple example of sets:

Example 1 (Intervals on a real line)**.** Let $(-1, 0]$ and $[0, 1)$ be two intervals on a real line. $(-1, 0] \cup [0, 1) = (-1, 1)$, $(-1, 0] \cap [0, 1) = \{0\}$, $(-1, 0] - [0, 1) = (-1, 0)$.

Definition 1. *Two sets are disjoint if they have no common elements. Sets $A_1 \dots A_n$ form a partition of a set A if sets are pairwise disjoint (that is, $\forall i, j\ A_i \cap A_j = \emptyset$ and their union forms the whole set A.*

Note that the rule of inclusion/exclusion simplifies greatly when sets are disjoint: in this case, the sum of the sizes of the disjoint sets is exactly the size of their union. This is used in probability theory.

Some subset relations:

1) $A \cap B \subseteq A$
2) $A \subseteq A \cup B$
3) Transitivity: if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

To prove that $A \subseteq B$ show that any element $x \in A$ is also $\in B$. Proof style: "suppose x is in A. Now show that $x \in B$." Now use logic of the definitions.

Now can prove properties of sets such as DeMorgan, by using "suppose x is in the left side... show that x is in the right side".

To prove something does not hold, find a counterexample.

Example 2. Show that it is not true that for all A, B, C, $(A - B) \cup (B - C) = A - C$. To prove this, find a counterexample, that is, sets A, B, C for which this does not hold. Let $A = \{1, 2\}$, $B = \{2\}$ and $C = \{1\}$. Then $A - C = \{2\}$, $A - B = \{1\}$, $B - C = \{2\}$ and the union is $\{1, 2\}$. Alternatively, think of an element in the LHS that is not in $A - C$: in this case, such an element is some element not in A.

6 Boolean algebra

As you have noticed, the algebra of sets is very similar to the algebra of propositions. This is because they are all examples of boolean algebras.

Definition 2. *A Boolean algebra is a set B together with two operations, generally denoted $+$ and $\cdot$, such taht for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following properties hold:*

- *Commutative laws: $a + b = b + a$ and $a \cdot b = b \cdot a$.*
- *Associative laws: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.*
- *Distributive laws: $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + c = (a + c) \cdot (b + c)$ (recall that the second one does not hold for the normal arithmetic $+$ and $\cdot$).*
- *Identity laws: $a + 0 = a$ and $a \cdot 1 = a$*
- *Complement laws: for each a there exists an element called negation of a and denoted $\bar{a}$ such that $a + \bar{a} = 1$, $a \cdot \bar{a} = 0$.*

In the case of propositional logic, 0 is F, 1 is T and there are no other elements, so it is sufficient to say that $\bar{T} = F$ and $\bar{F} = T$ (in that setting, $\neg$ is used for complementation). In set theory, 0 and 1 are $\emptyset$ and the universe U, respectively, and negation of every set is its complement.

Now, properties of Boolean algebras such as DeMorgan's law can be derived from these axioms.