CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 11

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Recall from the last class that predicates are essentially propositions with parameters (that is, they become true or false depending on the values of the parameters), and quantifiers $\forall x$ ($\exists x$) mean that the formula under the quantifier is true on all (respectively, some) element(s) from the domain. Variables that occur under quantifiers (that is, as $\forall x$ or $\exists x$) are called *bound* variables; if a variable is not bound, then it is called a *free variable.* The truth value of a first-order formula is defined when either free variables are given some specific values, or all variables are bound.

Lets look at some examples of first-order formulas using the $Parent(x, y)$ relation, saying that x is a parent of y.

- $\exists y\ Parent(x, y) \wedge Parent(z, y)$ says x and y have a common child. Here, x and z are free variables, and y is a bound variable.
- $\exists y\ Parent(x, y) \wedge (\exists y\ Parent(z, y))$ says that both x and y have children. Here, y in the first occurrence of $Parent$ is a different variable from y in the second occurrence. We talk about the *scope* of a quantifier to mean part of the formula in which the variable y is the variable mentioned in the quantifier. Usually, a scope of a quantifier starts from the place it occurs in the formula and continues until either there is another quantifier with the same variable name or the parentheses started before the quantifier are closed.

  Altough it is technically allowed to reuse the variable names, and it is sometimes useful in practice (think of a variable name, in software, denoting an allocated chunk of memory, and reusing the name as reusing the memory), for readability it is better to use different names for different variables. For example, the formula above has an equivalent but more readable version $\exists y\ Parent(x, y) \wedge \exists u\ Parent(z, u)$. Note that this would allow us to use u and y together in a predicate later in the formula, if we wish to.

- $\exists y\ Parent(x, y) \wedge Parent(y, z)$ says that x is a grandparent of z
- $\forall x \exists y Parent(y, x)$ says that for everybody somebody is his/her parent. This would be true about most of reasonable domains. However, changing the order of quantifiers here we obtain a formula with a very different meaning: $\exists y \forall x Parent(y, x)$ says that y is everybody's parent, which is not likely to be true.

4.1 Application to databases

A major application of predicate logic to computer science is in the database theory. Consider a database of all students at MUN, which has student personal information together with the names of their programs, courses they are taking / completed with their grades and so on. It is possible to request only certain part of information from the database: for example, we can write $Current(x, y)$ to mean a student x is currently taking the course y. If you think of the database as a table, think about columns "name", "semester" and "course": then $Current(x, y)$ is matching all lines of the table with semester being the current semester, student being x and the course being y. Another database predicate, let us call it $Done(x, y)$, could be true on all pairs x, y such that the student x is taking a course y. Usually, if a database query (formula) has free variables, then the query returns the list of all combinations of values (e.g., pairs of values if there are two free variables) on which the formula is true.

Now, consider the following formulas talking about these database predicates. A formula $Done(x, cs2711) \wedge Current(x, cs2742)$ would hold for x being a student who took cs2711 before cs2742. So such a query would return a list of all students who have done cs2711 before cs2742. If we just want to know if there exists such a student, we would ask whether $\exists x\ Done(x, cs2711) \wedge Current(x, cs2742)$. This query would return true if such x is found. Another query, $\forall x \exists y\ Current(x, y)$ states that every person in the database is a student currently taking courses. $\forall y \exists x\ Current(x, y)$ states that there is a course that is currently being taken by all students (not very likely to happen).

4.2 Multiple quantifiers

As you might have noticed with the previous example about "all students taking a course / course taken by all students" changing the order of quantifiers in a formula completely changes its meaning. Note that it only applies to changing order of quantifiers of different types; changing the order of two $\exists$ quantifiers next to each other would not change the meaning.

For any integer x there is an integer y such that $x + y = 5$. This is not the same as "there exists y such that for all x $x + y = 5$"!

To make it easier to remember, think of the following English phrase:

Everybody loves somebody

There are two ways of reading it, corresponding to two different orders of quantifiers:

> For every person, there is somebody this person loves (e.g, every person loves their mother).

> There exists a person whom everybody loves (e.g., everybody loves Elvis Presley).

So, when you are translating from English a sentence with alternating quantifiers, think: is the meaning "mother" or is the meaning "Elvis?"