

CS 2742 (Logic in Computer Science)

Lecture 6

Antonina Kolokolova

September 21, 2009

2.1 More on DNFs and CNFs

Recall that a formula is in the CNF (conjunctive normal form) if it is a \wedge of \vee s of literals (variables or their negation.) It is in the DNF (disjunctive normal form) if it is a \vee of \wedge of literals. Let us do a larger example of constructing a CNF and DNF from a truth table.

First, note that every truth assignment can be encoded as a propositional formula which is true just on that assignment and false everywhere else. In order to make such a formula, take \wedge of all variables that are true in the truth assignment and negations of all variables that are false in that truth assignment. Since every variable is either true or false, every variable will occur exactly once in this conjunction, either positively or negatively. For example, a truth assignment $p = T, q = F, r = T$ can be represented by a formula $p \wedge \neg q \wedge r$. This formula will be true when $p = T, q = F$ and $r = T$ and false everywhere else.

This representation of truth assignments by formulas is our building block for constructing DNFs and CNFs from a truth table. A DNF says that one of the truth assignments listed is true, where the assignments listed are all the satisfying assignments of a formula. A CNF is just a negation of a DNF formula listing falsifying truth assignments, simplified using DeMorgan's law to place negations on variables.

Let us look at a larger example for constructing CNFs and DNFs, equivalent to formulas (given by truth tables). That is, the CNF, the DNF and original formula, although they look quite different, share the same set of variables and are satisfied by exactly the same truth assignments. Their truth tables are identical.

Example 1. Last time we used a truth table of a propositional formula $(p \rightarrow q) \wedge q \rightarrow p$ for constructing DNFs and CNFs. Now let us add one more variable: consider a formula $((p \rightarrow q) \wedge q \rightarrow p) \wedge (\neg q \vee \neg r)$.

r	p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$(p \rightarrow q) \wedge q \rightarrow p$	$\neg q$	$\neg r$	$(\neg q \vee \neg r)$	$((p \rightarrow q) \wedge q \rightarrow p) \wedge (\neg q \vee \neg r)$
T	T	T	T	T	T	F	F	F	F
T	T	F	F	F	T	T	F	T	T
T	F	T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	F	T	T
F	T	T	T	T	T	F	T	T	T
F	T	F	F	F	T	T	T	T	T
F	F	T	T	T	F	F	T	T	F
F	F	F	T	F	T	T	T	T	T

There are three F in the last column and five T, so our CNF will have 3 clauses (that is, it will be the \wedge of three \vee), and our DNF will have 5 terms.

So, to construct a CNF for this formula, take

$$\neg((r \wedge p \wedge q) \vee (r \wedge \neg p \wedge q) \vee (\neg r \wedge \neg p \wedge q))$$

After simplification (applying DeMorgan's law and double negation law), this becomes

$$(\neg r \vee \neg p \vee \neg q) \wedge (\neg r \vee p \vee \neg q) \wedge (r \vee p \vee \neg q)$$

You can check yourself that this formula is CNF and is indeed equivalent to $((p \rightarrow q) \wedge q \rightarrow p) \wedge (\neg q \vee \neg r)$; that is, it has the same truth table. A DNF for this formula is:

$$(r \wedge p \wedge \neg q) \vee (r \wedge \neg p \wedge \neg q) \vee (\neg r \wedge p \wedge q) \vee (\neg r \wedge p \wedge \neg q) \vee (\neg r \wedge \neg p \wedge \neg q)$$

Note that between the CNF and DNF all 8 truth assignments are mentioned: the falsifying ones in the CNF and satisfying in DNF.

3 Resolution.

Recall that a formula is in the CNF (conjunctive normal form) if it is a \wedge of \vee s of literals (variables or their negation.)

In this lecture we will talk about proving (or, actually, finding contradictions) statements that are in this special form.

Definition 1 (Resolution rule). : *Given two clauses of the form $C \vee x$ and $(D \vee \neg x)$, where C and D are (possibly empty) disjunction of variables, can derive a (possibly empty) clause $(C \vee D)$.*

That is,

$$(C \vee x) \wedge (D \vee \neg x) \rightarrow (C \vee D)$$

where $C = (l_1 \vee \dots \vee l_k)$ and $D = (l'_1 \dots l'_{k'})$ for some literals. If there is a repeated literal, only write it once.

Note that you may end up deriving an “empty” clause, that is, a \vee of zero literals. But the only way this could happen is when two clauses being resolved are (x) and $(\neg x)$. But $x \wedge \neg x$ is always false, a contradiction. So deriving an empty clause proves that the original formula was a contradiction (which is what we usually want to show).

Example 2. Consider the following statements:

p : it is sunny

q : the weather is good

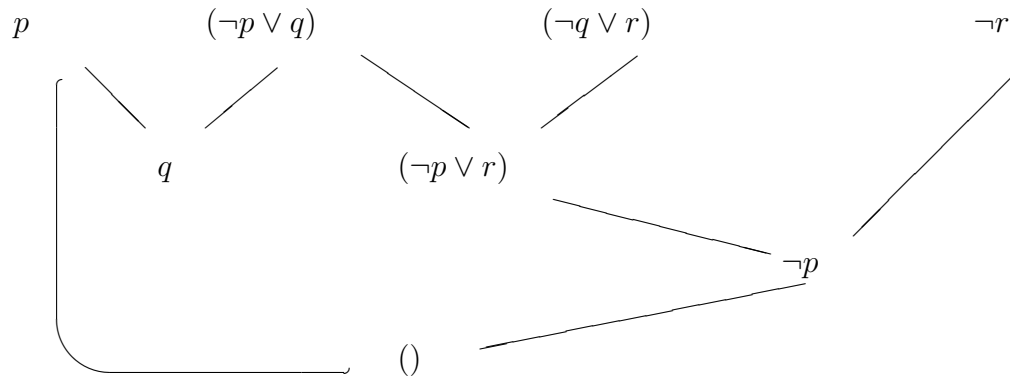
r : I spend the day outside

Now consider the following argument:

$$\begin{array}{l} p \\ p \rightarrow q \\ q \rightarrow r \\ \therefore r \end{array}$$

We want to prove that this is a valid argument, that is, $p, p \rightarrow q$ and $q \rightarrow r$ together imply r (this in general is called a *transitivity* law). Resolution proof method allows us to find contradictions: so we represent this problem as a contradiction $p \wedge (p \rightarrow q) \wedge (q \rightarrow r) \wedge \neg r$. Note that here again we are using the fact that the negation of an implication (in this case, premises imply the conclusion) is a conjunction of premises and negation of the conclusion.

Resolution works with CNF formula, so the first step is to convert the formula above into CNF. In this case, it is very easy: just apply the definition of implication to the second and third clause to obtain $p \wedge (\neg p \vee q) \wedge (\neg q \vee r) \wedge \neg r$.



Puzzle 6. Prove that in a big city like Toronto with more than a million people there would be at least two people with the same number of hairs on their heads. Assume that the number of hairs on somebody’s head cannot be more than 300,000.