CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 5

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1.1 More about implications

In Lecture 4 we had the following knights-and-knaves puzzle:

A said: "If I am a knight I'll eat my hat!". Show that A will eat his hat.

Note that this statement is an implication. Lets set p: "A is a knight" and q: "A will eat his hat". Then what A said is $p \to q$. Now, consider the truth table for the statement saying that what A said is truth if and only if A is a knight (that is, $(p \to q) \leftrightarrow p$.)

p	q	$p \to q$	$(p \to q) \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

So the only situation which is possible (that is, A is a knight and he told the truth or A is a knave and he lied) is when A is a knight and he told the truth. And since what he said is true, and the left hand side of the implication (that is, p) is true, q also has to be true. So A will eat his hat.

Let us look what happens when A is a knave. Then what he said must be false. But the only time $p \to q$ is false is when p is true and q is false (that is, $\neg(p \to q) \iff (p \land \neg q)$, you can check by the definition of implication and the DeMorgan's law that this holds). So here is where the contradiction comes: the only time the implication could be false (that is, uttered by a knave) is when its left-hand-side is true (that is, A is a knight).

One of the main things to remember about the implication is that **falsity can imply anything!**. That is, if pigs can fly, then 2+2=5, and also if pigs can fly then 2+2=4.

Both of these are true implications, provided that pigs cannot fly. No matter what kind of statement is q, the implication $F \to q$ is always true.

A brilliant example that shows that falsity can imply anything was presented by the famous logician Bertrand Russell (author of the "Russell's paradox" that we will see later in the course.)

Example 1. Bertrand Russell: "If 2+2=5, then I am the Pope".[1]

Proof. If 2+2=5 then 1=2 by subtracting 3 from both sides.
Bertrand Russell and Pope are two people.
Since 1=2, Bertrand Russell and Pope are one person.

□

Note that the steps of this proof are perfectly fine logically. Every line follows from the previous line correctly. The strange conclusion exclusively from 2+2=5 being a false statement.

1.2 Valid and invalid arguments

Bertran Russell once said "if 2+2=5, then I am the Pope". His proof was a valid argument (that is, every step logically followed from the previous); but since his premise "2+2=5" was false, it is no wonder that his conclusion was false as well.

If Socrates is a man, then Socrates is mortal
Socrates is a man
$\therefore$ Socrates is mortal

Terminology: the final statement is called *conclusion*, the rest are *premises*. The symbol $\therefore$ reads as *therefore*. An argument is *valid* if no matter what statements are substituted into variables, if all premises are true then the conclusion is true.

The valid form of agument is called *rules of inference*. The most known is called *Modus Ponens* ("method of affirming"). Its contrapositive is called *Modus Tollens* ("method of denying"):

Modus Ponens	*Modus Tollens*
If p then q	If p then q
p	$\neg q$
$\therefore q$	$\therefore \neg p$

[1] According to other sources, the statement Bertrand Russell was proving was "if 1+1=1, then I am the Pope". In this case, the first line can be omitted.

This is another way to describe "proof by contrapositive". Similarly we can write the proof by cases, by contradiction, by transitivity and so on. They can be derived from the original logic identities. For example, modus ponens becomes $((p \to q) \land p) \to q$.

Example 2. The general form of the proof by cases above can be written as follows:

$p \lor q$	"n is even or n is odd"
$p \to r$	"if n is even then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "
$q \to r$	"if n is odd then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "
$\therefore r$	"therefore $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "

To show that an argument is invalid give a truth table where all premises are true and the conclusion is false. That is, show that a propositional formula of the form $\land$(of premises) $\to$ conclusion is not a tautology.

Example 3. For example, although modus ponens is valid, the following is not a valid inference:

$p \to q$
q INVALID ARGUMENT!
$\therefore p$

We can write it as $((p \to q) \land q) \to p$ and show that this is not a tautology using a truth table.

p	q	$p \to q$	$(p \to q) \land q$	$(p \to q) \land q \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	**F**
F	F	T	F	T

The truth table showed us a situation when both premises $(p \to q)$ and q are true, but the conclusion p is false. Therefore, $((p \to q) \land q) \to p$ is not a tautology and thus the argument based on it is not a valid argument.

2 Normal forms of propositional formulas.

Any formula has an equivalent one in a normal form. In computer science, the two normal forms we are interested in are *conjunctive normal form (CNF)* and *disjunctive normal form*

(DNF). The first one is a $\wedge$ of $\vee$s, where $\vee$ is over variables or their negations (*literals*), the second is a $\vee$ of $\wedge$s of literals. A $\vee$ of literals is also called a *clause*, and a $\wedge$ of literals a *term*.

Example 4.

- $\neg p$, x, s are examples of literals, whereas $\neg\neg p$ or $(x \vee y)$ is not a literal
- $(x \vee \neg y \vee z)$, $(\neg p)$ are clauses,
- $(x \wedge \neg y \wedge z)$, $(\neg p)$ are terms.
- $(x \vee \neg z \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg y)$ is a CNF.
- $(x \wedge z) \vee (\neg y \wedge z \wedge x) \vee (\neg x \wedge z)$ is a DNF.
- $(x \wedge \neg(y \vee z) \vee u)$ is neither a CNF nor a DNF, but it is equivalent to the following DNF: $(x \wedge \neg y \wedge \neg z) \vee u$.

Why do we need CNFs and DNFs? Other than the fact that they are convenient normal forms, they are very useful for constricting formulas given a truth table. In particular, DNFs can be constructed from truth table by taking a disjunction (that is, $\vee$) of all satisfying truth assignments and CNFs by taking a conjunctio ($\wedge$) of negations of falsifying truth assignments

Example 5. Recall the truth table in example 3 above. Suppose that you are given just the first two and the last column. How do you construct a formula that would have such a truth table?

Let us start by constructing a DNF formula. It would say, basically, "either the first line satisfies me, or the second, or the last". We write it as $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$. Each term here fully describes one satisfying assignment, and the whole formula is true iff $(p \to q) \wedge q \to p$ is. Now, to construct a formula in CNF we make it say "I am satisfied by neither of the following assignments". One way to write it is as a negation of a DNF formula listing all falsifying assignments. In this example there is just one falsifying assignment, so the formula becomes $\neg(\neg p \wedge q)$. By DeMorgan's law, this is equivalent to $(\neg\neg p \vee \neg q)$, and finally $(p \vee \neg q)$. If there were more than one falsifying assignment, here we would have several of clauses with $\wedge$ between them.

This construction also shows us that for any formula there is an equivalent CNF and an equivalent DNF formula (these are the formulas constructed from the truth table of the original formula). In particular, if the formula is a tautology, its DNF and CNF consist of just $(p \vee \neg p)$ for an arbitrary p, and if it is a contradiction, of $(p \wedge \neg p)$.