CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 4

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Absorption laws: $p \vee (p \wedge q) \iff p \wedge (p \vee q) \iff p$.

1.1 Conditional statements

Conditional statements are ones of the form "if p then q", $p \to q$. Recall that we logically define $p \to q \iff (\neg p \vee q)$.

We use the following terminology when talking about conditional statements:

1) *Contrapositive* of $p \to q$ is $\neg q \to \neg p$. True whenever the original implication is.

   Proof. Recall that $(p \to q) \iff (\neg p \vee q)$. Now,

$$\begin{aligned} & \neg q \to \neg p \\ \iff & (\neg\neg q \vee \neg p) && \text{Definition of } \to \\ \iff & (q \vee \neg p) && \text{Double negation} \\ \iff & (\neg p \vee q) && \text{Commutativity} \iff (p \to q) \qquad \text{Definition of } \to \end{aligned}$$

   Thus, a contrapositive of an if-then statement is logically equivalent to the original statement. □

2) *Converse* and *inverse*: $q \to p$ and $\neg p \to \neg q$. Contrapositives of each other, can have a different truth value from $p \to q$.

3) *Sufficient* condition: p is sufficient for q if $p \to q$. The "if" part of if-then.

4) *Neccessary* condition: p is necessary for q if $\neg p \to \neg q$, that is, $q \to p$. The "then" part of "if-then".

5) If and only if (p iff q, $p \leftrightarrow q$) means $(p \to q) \wedge (q \to p)$.

Example 1. Consider the sentence "if n is divisible by 4, then n is divisible by 2" (we will use the notation $n|4$ to mean n is divisible by 4). This is an if-then statement. Its contrapositive is "if $n \not| 2$ then $n \not| 4$. That is, if n is an odd number then it is definitely not divisible by 4.So $n|4$ is sufficient for $n|2$ (if n is divisible by 4, it is sufficient for n to be divisible by 2). On the other hand, $n|2$ is necessary for $n|4$.

1) *Direct proof*: show that if p is true directly.

2) *Proof by contrapositive*: instead of $p \to q$ prove $\neg q \to \neg p$.

   Lemma 1. *If n^2 is even, then n is even.*

   Proof. We will show this by showing that if n is odd, then n^2 is odd. If n is odd, then $n = 2k+1$ for some k. Then $(2k+1)^2 = 2(2k^2+2k)+1$, which is an odd number. This proves that if n is odd, then n^2 is odd, thus proving the contrapositive of if n^2 is even then n is even, and so proving the statement "if n^2 is even then n is even" itself. □

3) *Proof by contradiction:* to show that p is true, show that $\neg p \to F$.
   It is easy to show that $(\neg p \to F)$ is logically equivalent to p: just note that $(\neg\neg p \vee F) \iff (p \vee F) \iff p$ by applying the definition of implication followed by the double negation law followed by the identity law.

   Theorem 1. *$\sqrt{2}$ is irrational.*

   Proof. Recall that a number is called *rational* if it can be represented as an (irreducible) fraction of two integers. Assume, for the sake of contradiction, that $\sqrt{2}$ is rational: that is, there are integers m and n which do not have any common divisors > 1 such that $\sqrt{2} = m/n$.

   - If $\sqrt{2} = m/n$ then $(\sqrt{2})^2 = m^2/n^2$.
   - From here, $2n^2 = m^2$, which means that m^2 is even.
   - By the lemma above, then m is even, so $m = 2k$ for some k.
   - Then $m^2 = 4k^2$. So $2n^2 = 4k^2$, and, dividing by 2, $n^2 = 2k^2$. So n^2 is even.
   - Using the lemma again, conclude that n is even.
   - So both n and m are even, but we assumed that m and n do not have a non-trivial common divisor. This is a contradiction.

   □

4) *Proof by cases:* to show that p is true, prove $(q \to p) \wedge (\neg q \to p)$.

Lemma 2. *For any natural number n, $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$*

Here, $\lfloor k \rfloor$ is a *floor* of a (real) number, defined to be the largest integer smaller than or equal to k. For example, $\lfloor 5/2 \rfloor = 2$, and $\lfloor 4/2 \rfloor = 2$. Similarly, a *ceiling* of a number is the smallest integer larger than or equal to that number. So $\lceil 5/2 \rceil = 3$ and $\lceil 4/2 \rceil = 2$. For an integer, both its floor and its ceiling are equal to that integer itself; for a non-integer, the floor is is rounded-down value and the ceiling is rounded up.

Proof. **Case 1: n is even**. Then $\lfloor (n+1)/2 \rfloor = n/2 = \lceil n/2 \rceil$.
Case 2: n is odd. Then $\lfloor (n+1)/2 \rfloor = (n+1)/2 = \lceil n/2 \rceil$. □

Puzzle 4. A from the island of knights and knaves said: "If I am a knight, then I'll eat my hat!". Prove that A will eat his hat.