CS 2742 (Logic in Computer Science)
Lecture 3

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1.1 Logical equivalences

Recall the puzzle from the previous class: on some island, there are knights (who always tell the truth) and knaves (who always lie). You meet two islanders (call them A and B) and hear the first one say "at least one of us is a knave". Can you tell whether the islanders are knights or knaves and which islander is which?

We solve this puzzle using a truth table. Take p :"A is a knight" and q : "B is a knight.". Then the sentence "At least one of us is a knave" is translated as $(\neg p \vee \neg q)$, since being a knave is the negation of being a knight. Now, we want to know when the truth value of this sentence $(\neg p \vee \neg q)$ is the same of the truth value of p: that is, if A is a knight, then what he said must be true, and if A is a knave, then what he said must be false. Let us introduce the notation $\iff$ to mean *logical equivalence* (that is, two formulas having the same truth values for any truth assignment to their variables). Then the last condition becomes $p \iff (\neg p \vee \neg q)$.

We represent this as a truth table as follows:

p	q	$(\neg p \vee \neg q)$	$p \iff (\neg p \vee \neg q)$
T	T	F	F
T	F	T	T
F	T	T	F
F	F	T	F

As you can see, the only scenario when what A says corresponds correctly with A's being a knight/knave is the second line: that is, when A is a knight and B is a knave.

Definition 1 *We say that two formulas are* logically equivalent *($A \iff B$) if they have the same truth value under any truth assignment.*

That is, in a truth table the columns of logically equivalent formulas are identical. This is a semantic notion of equivalence. It can also be defined syntactically: $p \leftrightarrow q$ defined as $(p \rightarrow q) \wedge (q \rightarrow p)$. You can check that this formula holds if and only if p and q have the same value.

A useful property of logically equivalent formulas, called *substitution*, is that if in any formula you replace a subformula by another that is logically equivalent to it, then the value of the whole formula would not change. For example, $p \wedge (q \vee \neg q)$ is logically equivalent to $(p \wedge T)$, which is in turn equivalent to p (can check this using a truth table). So if in a formula there is an occurrence of $p \wedge (q \vee \neg q)$ it can be safely replaced with just p, thus simplifying the formula.

1.2 Logical identities

Now that we have a notion of logical equivalences we can talk about a few identities in propositional logic. We will list them as pairs of equivalent formulas.

Name	$\wedge$-version	$\vee$-version
Double negation	$\neg\neg p \iff p$	
DeMorgan's laws	$\neg(p \wedge q) \iff (\neg p \vee \neg q)$	$\neg(p \vee q) \iff (\neg p \wedge \neg q)$
Commutativity	$(p \wedge q) \iff (q \wedge p)$	$(p \vee q) \iff (q \vee p)$
Associativity	$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$	$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$
Distributivity	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
	$p \wedge F \iff F$	$p \vee T \iff T$
Idempotence	$p \wedge p \iff p$	$p \vee p \iff p$
Absorption	$p \wedge (p \vee q) \iff p$	$p \vee (p \wedge q) \iff p$

Notice again (as working our way to boolean algebras) that many of these identities behave just like algebraic and arithmetic identities, with $\wedge$ behaving like $\times$, $\vee$ like $+$, T like 1 and F like 0. For example, commutativity and associativity laws are the same as for numbers: $(3+2)+5 = 3+(2+5)$. One notable exception is that with numbers there is just one form of distributed law, namely the $\wedge$ form $(a \times (b+c) = (a \times b) + (a \times c)$, and the $\vee$ form does not hold $(a + b \times c) \neq (a \times b) + (a \times c))$, whereas in logic both forms are true.

1.3 Simplifying propositional formulas.

Now we can apply these identities to simplify propositional formulas.

Example 1

$$\begin{aligned}
& (p \wedge q) \vee \neg(\neg p \vee \neg q) & \\
\iff & (p \wedge q) \vee (\neg\neg p \wedge \neg\neg q) & \text{Apply DeMorgan's} \\
\iff & (p \wedge q) \vee (p \wedge q) & \text{Double Negation (twice)} \\
\iff & (p \wedge q) & \text{Idempotence}
\end{aligned}$$

Notice that the logic identities are stated only for the logical connectives $\wedge, \vee, \neg$. In order to deal with $\rightarrow$ and $\iff$ we use their definitions: for example, $\phi \rightarrow \psi$ becomes $\neg\phi \vee \psi$.

Example 2

$$\begin{aligned}
& p \leftrightarrow ((q \wedge \neg r) \rightarrow q) & \\
\iff & p \leftrightarrow (\neg(q \wedge \neg r) \vee q) & \text{Definition of } \rightarrow \\
\iff & p \leftrightarrow ((\neg q \vee \neg\neg r) \vee q) & \text{DeMorgan} \\
\iff & p \leftrightarrow ((\neg\neg r \vee \neg q) \vee q) & \text{Commutativity} \\
\iff & p \leftrightarrow (\neg\neg r \vee (\neg q \vee q)) & \text{Associativity (dropping parentheses)} \\
\iff & p \leftrightarrow (\neg\neg r \vee T) & \text{Definition of T} \\
\iff & p \leftrightarrow T & \text{Identity} \\
\iff & p & \text{Because } \leftrightarrow \text{ is an equivalence}
\end{aligned}$$

The last step could be done more formally as follows:

$$\begin{aligned}
& p \leftrightarrow T & \\
\iff & (p \rightarrow T) \wedge (T \rightarrow p) & \text{Definition of } \leftrightarrow \\
\iff & (\neg p \vee T) \wedge (\neg T \vee p) & \text{Definition of } \rightarrow \\
\iff & (\neg p \vee T) \wedge (F \vee p) & \text{Definition of } F \\
\iff & T \wedge (F \vee p) & \text{Identity} \\
\iff & (F \vee p) & \text{Identity} \\
\iff & p & \text{Identity}
\end{aligned}$$

Puzzle 3 What is the value of $2 + 2 = 4$?