CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 25

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7.1 Variants of induction

You have seen already the Well-Ordering Principle, which can be considered an (equivalent) variant of induction. In this lecture we will look at another (also equivalent, although looking more powerful) variant of induction, called *strong* (or sometimes *complete*) induction. Here, instead of assuming that $P(i)$ holds for just one preceding element i, we assume that it holds for all elements from the base case up to (but not including) k, and then proceed with this stronger assumption to proving $P(k)$. We will prove the equivalence of the three principles later.

Definition 1 (Strong induction)**.** *Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:*

1) ***Base case**: for some $b \geq a$, $\forall a \leq c \leq b, P(c)$ is true.*

2) ***Induction step:** $(\forall i, b \leq i < kP(i)) \to P(k)$*

Then the statement

$$\text{for all integers } n \geq a, \ P(n)$$

is true.

Example 1. Here is another way of solving the 3c and 5c coins problem, this time using strong induction. Recall that the goal is to prove that $\forall n \geq 8, \exists i, j \geq 0 \ n = 3i + 5j$.
Proof: Let $P(n)$ be $\exists i, j \geq 0 \ n = 3i + 5j$, as before.
`Base case`: This time, there are three base cases, $n = 8 = 3 \cdot 1 + 5 \cdot 1$, $n = 9 = 3 \cdot 3 + 5 \cdot 0$, and $n = 10 = 3 \cdot 0 + 5 \cdot 2$.
`Induction hypothesis` Assume that $\forall m, 8 \leq m < k$, $\exists i, j \geq 0 m = 3i + 5j$.
`Induction step`. As in the proof with well-ordering, consider $k - 3$. If $k - 3 \geq 8$, then

there are i, j such that $k-3 = 3i+5j$ and so $k = 3(i+1)+5j$. Otherwise, k must be one of the three base cases $8, 9$ or 10, for which we know the corresponding i and j.

In this example, we made use of two things: first, strong induction allowed us to talk about the value of $k-3$ as opposed to just $k-1$. Second, we explicitly used base cases.

The following is a classical example of using strong induction. It shows how it is applicable in cases where we do not know beforehead which elements between the base case and k we need to use.

Example 2 (Divisibility by prime)**.** Show that for every natural number $n \geq 2$, n is divisible by a prime number.
Proof: Let $P(n)$ be a predicate $\exists p \in \mathbb{N}, 2 \leq p < n,\ p|n \wedge \forall q, q \not| p$. Here, the notation $q \not| p$ ("q does not divide p") means that there is no such integer r that $p = qr$.
`Base case:` 2 is a prime, so it is divisible by itself.
`Induction hypothesis:` Assume that for all numbers i, $2 \leq i < k$, i is divisible by a prime number p (that is, $\exists p\ geq 2$ such that $p|i$).
`Induction step:` Look at a number k. If k is prime, done, since k is divisible by itself. If it is not, then by definition of a number being not prime $\exists a, b \geq 2$, $k = ab$. Take a to be our i from induction hypothesis. By induction hypothesis, there is a prime numbers $p \geq 2$ such that $p|a$ Since the division relation is transitive, $p|k$. Here, we don't even need to use the induction hypothesis for b; we would if we were proving the Unique Factorization Theorem that says that any number can be represented as a product of powers of primes.

Here, we relied heavily on having the strong induction hypothesis, because a and b can be any numbers between 2 and $k/2$.