CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 23

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8 Zorn's lemma, axiom of choice, well-ordering principle and induction

A chain in a partial order is a totally ordered subset (i.e., $\{2, 6, 12\}$ in our division example). Length of chain is the number of elements -1.

Zorn's lemma: every partially ordered set in which every chain has a upper bound contains at least one maximal element. Here, note that the upper bound is an element of the partially ordered set, but does not have to be an element of the totally ordered subsets that are chains.

Well-ordering principle: let S contain one or more integers all of which are greater than some fixed integer. Then S has a least element. Restating: every non-empty set of positive integers contains a smallest element.

Remember that natural numbers in set theory are defined in such a way that this principle holds (using axiom of choice).

Zorn's lemma, axiom of choice and well-ordering principles are equivalent.

Axiom of choice: "out of any given collection of bins it is possible to select exactly one element from each bin, even if there are infinitely many bins and no rule by which to select a bin".

Here is an example of applying well-ordering principle.

Example 1. Show that every amount of change $n \geq 8$ can be paid with only 3c and 5c coins.
Proof: Suppose, for the sake of contradiction, that there are some values of $n \geq 8$ such that it is not possible to pay n with 3c and 5c coins. Take the set of all such values. Since all

of them are natural numbers, there is, by well-ordering principle, a minimal element in this set; let's call it k. Now, consider number $k-3$. There are two possibilities. First, it can be that $k-3<8$. Since $k \geq 8$ the only choices for k are $8, 9$ and 10. But $8=3+5$, $9=3*3$ and $10=5*2$, so in all these cases k is representable by a sum of 3s and 5s. So it should be that $k-3 \geq 8$. But then, $k-3$ is not representable as a sum of 3s and 5s either (otherwise if $k-3=3i+5j$, then $k=3(i+1)+5j$.) But this contradicts the fact that k was the *smallest* such element, given to us by the well-ordering principle. Therefore, every $n \geq 8$ is representable as a sum of 3s and 5s.

8.1 Induction

The statement of mathematical induction is a contrapositive to the well-ordering principle:

Definition 1. *Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:*

1) $P(a)$ is true. (called ***base case****)*

2) For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. (called ***induction step****).*

Then the statement

$$\textit{for all integers } n \geq a,\ P(n)$$

is true.

Alternatively, the axiom of induction can be written as follows:

$$\text{for all predicates } P, (P(0) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n)$$

Structure of a proof by induction:

1) **Predicate** State which $P(n)$ you are proving as a function of n.

2) **Base case:** Prove $P(a)$.

3) **Induction hypothesis:** State "Assume $P(k)$ holds" explicitly.

4) **Induction step:** Show how $P(k) \rightarrow P(k+1)$. That is, assuming $P(k)$ derive $P(k+1)$.

Example 2. Show that for all $n \geq 0$, $0 + 1 + \cdots + n = n(n+1)/2$. This is a classical example of application of math. induction.
Proof: Predicate: $P(n) = 0 + 1 + \cdots + n = n(n+1)/2$.
Base case: $n = 0$, then $0 = 0 \cdot (1/2)$. Let's also check $n = 1$: $0 + 1 = 1 = 1 \cdot (1+1)/2$
Induction hypothesis: Assume that for some $k \geq 0$ $0 + 1 + \cdots + k = k(k+1)/2$.
Induction step: Show that $P(k) \to P(k+1)$. Take $0 + 1 + \cdots + k + (k+1) = (0 + 1 + \cdots + k) + (k+1)$. By induction hypothesis, the sum in the first parentheses is $k(k+1)/2$. Now, $k(k+1)/2 + (k+1) = \frac{k(k+1)+2(k+1)}{2} = (k+2)(k+1)/2 = (k+1)(k+2)/2$, which is exactly the right hand side of $P(k+1)$.
Therefore, by induction, $\forall n \geq 0$, $0 + 1 + \cdots + n = n(n+1)/2$.

Note that in this case, the calculations would be slightly simpler if we would state the induction hypothesis and induction step as "Assume $P(k-1)$, prove $P(k)$". This is a valid argument, and is often used, as long as $k-1$ satisfies the restriction on n (in this case, $k - 1 \geq 0$).

8.2 Sum and product notation.

Can write $0 + 1 + \cdots + n = \Sigma_{i=1}^{n} i$. In general, can have any arithmetic formula in the sum, e.g., $\Sigma_{i=1}^{n} \frac{i^2-10}{\sqrt{i}+5}$. Also, can write a product as $0 \cdot 1 \cdot \cdots \cdot n = \Pi_{i=1}^{n} i$.

Puzzle 1. What is wrong with the following induction proof of: "All horses are white"?

1) Let $P(n)$ be: any n horses are white.

2) Base case: 0 horses are white.

3) Ind. hyp.: if n horses are, so are $n + 1$.