

# CS 2742 (Logic in Computer Science) – Fall 2008

## Lecture 16

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Before continuing, let us look at a simple example of sets:

**Example 1** (Intervals on a real line). Let  $(-1, 0]$  and  $[0, 1)$  be two intervals on a real line.  $(-1, 0] \cup [0, 1) = (-1, 1)$ ,  $(-1, 0] \cap [0, 1) = \{0\}$ ,  $(-1, 0] - [0, 1) = (-1, 0)$ .

**Definition 1.** *Two sets are disjoint if they have no common elements. Sets  $A_1 \dots A_n$  form a partition of a set  $A$  if sets are pairwise disjoint (that is,  $\forall i, j A_i \cap A_j = \emptyset$  and their union forms the whole set  $A$ ).*

Note that the rule of inclusion/exclusion simplifies greatly when sets are disjoint: in this case, the sum of the sizes of the disjoint sets is exactly the size of their union. This is used in probability theory.

Some subset relations:

- 1)  $A \cap B \subseteq A$
- 2)  $A \subseteq A \cup B$
- 3) Transitivity: if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .

To prove that  $A \subseteq B$  show that any element  $x \in A$  is also  $\in B$ . Proof style: “suppose  $x$  is in  $A$ . Now show that  $x \in B$ .” Now use logic of the definitions.

Now can prove properties of sets such as DeMorgan, by using “suppose  $x$  is in the left side... show that  $x$  is in the right side”.

To prove something does not hold, find a counterexample.

**Example 2.** Show that it is not true that for all  $A, B, C$ ,  $(A - B) \cup (B - C) = A - C$ . To prove this, find a counterexample, that is, sets  $A, B, C$  for which this does not hold. Let  $A = \{1, 2\}$ ,  $B = \{2\}$  and  $C = \{1\}$ . Then  $A - C = \{2\}$ ,  $A - B = \{1\}$ ,  $B - C = \{2\}$  and the union is  $\{1, 2\}$ . Alternatively, think of an element in the LHS that is not in  $A - C$ : in this case, such an element is some element not in  $A$ .

## 6 Boolean algebra

As you have noticed, the algebra of sets is very similar to the algebra of propositions. This is because they are all examples of boolean algebras.

**Definition 2.** A Boolean algebra is a set  $B$  together with two operations, generally denoted  $+$  and  $\cdot$ , such that for all  $a$  and  $b$  in  $B$  both  $a + b$  and  $a \cdot b$  are in  $B$  and the following properties hold:

- *Commutative laws:*  $a + b = b + a$  and  $a \cdot b = b \cdot a$ .
- *Associative laws:*  $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- *Distributive laws:*  $(a + b) \cdot c = a \cdot c + b \cdot c$  and  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  (recall that the second one does not hold for the normal arithmetic  $+$  and  $\cdot$ ).
- *Identity laws:*  $a + 0 = a$  and  $a \cdot 1 = a$
- *Complement laws:* for each  $a$  there exists an element called negation of  $a$  and denoted  $\bar{a}$  such that  $a + \bar{a} = 1$ ,  $a \cdot \bar{a} = 0$ .

In the case of propositional logic, 0 is  $F$ , 1 is  $T$  and there are no other elements, so it is sufficient to say that  $\bar{T} = F$  and  $\bar{F} = T$  (in that setting,  $\neg$  is used for complementation). In set theory, 0 and 1 are  $\emptyset$  and the universe  $U$ , respectively, and negation of every set is its complement.

Now, properties of Boolean algebras such as DeMorgan's law can be derived from these axioms.