CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 6

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2.1 More about implications

In Lecture 4 we had the following knights-and-knaves puzzle:

A said: "If I am a knight I'll eat my hat!". Show that A will eat his hat.

Note that this statement is an implication. Lets set p: "A is a knight" and q: "A will eat his hat". Then what A said is $p \to q$. Now, consider the truth table for the statement saying that what A said is truth if and only if A is a knight (that is, $(p \to q) \leftrightarrow p$.)

p	q	$p \to q$	$(p \to q) \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

So the only situation which is possible (that is, A is a knight and he told the truth or A is a knave and he lied) is when A is a knight and he told the truth. And since what he said is true, and the left hand side of the implication (that is, p) is true, q also has to be true. So A will eat his hat.

Let us look what happens when A is a knave. Then what he said must be false. But the only time $p \to q$ is false is when p is true and q is false (that is, $\neg(p \to q) \iff (p \wedge \neg q)$, you can check by the definition of implication and the DeMorgan's law that this holds). So here is where the contradiction comes: the only time the implication could be false (that is, uttered by a knave) is when its left-hand-side is true (that is, A is a knight).

One of the main things to remember about the implication is that **falsity can imply anything!**. That is, if pigs can fly, then 2+2=5, and also if pigs can fly then 2+2=4.

Both of these are true implications, provided that pigs cannot fly. No matter what kind of statement is q, the implication $F \rightarrow q$ is always true.

A brilliant example that shows that falsity can imply anything was presented by the famous logician Bertrand Russell (author of the "Russell's paradox" that we will see later in the course.)

Example 1. Bertrand Russell: "If 2+2=5, then I am the Pope".[1]

Proof. If 2+2=5 then 1=2 by subtracting 3 from both sides.
Bertrand Russell and Pope are two people.
Since 1=2, Bertrand Russell and Pope are one person.

□

Note that the steps of this proof are perfectly fine logically. Every line follows from the previous line correctly. The strange conclusion exclusively from 2+2=5 being a false statement.

2.2 More on DNFs and CNFs

Recall that a formula is in the CNF (conjunctive normal form) if it is a $\wedge$ of $\vee$s of literals (variables or their negation.) It is in the DNF (disjunctive normal form) if it is a $\vee$ of $\wedge$ of literals. Let us do a larger example of constructing a CNF and DNF from a truth table.

First, note that every truth assignment can be encoded as a propositional formula which is true just on that assignment and false everywhere else. In order to make such a formula, take $\wedge$ of all variables that are true in the truth assignment and negations of all variables that are false in that truth assignment. Since every variable is either true or false, every variable will occur exactly once in this conjunction, either positively or negatively. For example, a truth assignment $p = T, q = F, r = T$ can be represented by a formula $p \wedge \neg q \wedge r$. This formula will be true when $p = T, q = F$ and $r = T$ and false everywhere else.

This representation of truth assignments by formulas is our building block for constructing DNFs and CNFs from a truth table. A DNF says that one of the truth assignments listed is true, where the assignments listed are all the satisfying assignments of a formula. A CNF is just a negation of a DNF formula listing falsifying truth assignments, simplified using DeMorgan's law to place negations on variables.

Let us look at a larger example for constructing CNFs and DNFs, equivalent to formulas (given by truth tables). That is, the CNF,the DNF and original formula, although they look

[1]According to other sources, the statement Bertrand Russell was proving was "if 1+1=1, then I am the Pope". In this case, the first line can be omitted.

quite different, share the same set of variables and are satisfied by exactly the same truth assignments. Their truth tables are identical.

Example 2. Last time we used a truth table of a propositional formula $(p \to q) \wedge q \to p$ for constructing DNFs and CNFs. Now let us add one more variable: consider a formula $((p \to q) \wedge q \to p) \wedge (\neg q \vee \neg r)$.

r	p	q	$p \to q$	$(p \to q) \wedge q$	$(p \to q) \wedge q \to p$	$\neg q$	$\neg r$	$(\neg q \vee \neg r)$	$((p \to q) \wedge q \to p) \wedge (\neg q \vee \neg r)$
T	T	T	T	T	T	F	F	F	F
T	T	F	F	F	T	T	F	T	T
T	F	T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	F	T	T
F	T	T	T	T	T	F	T	T	T
F	T	F	F	F	T	T	T	T	T
F	F	T	T	T	F	F	T	T	F
F	F	F	T	F	T	T	T	T	T

There are three F in the last column and five T, so our CNF will have 3 clauses (that is, it will be the $\wedge$ of three $\vee$), and our DNF will have 5 terms.

So, to construct a CNF for this formula, take

$$\neg((r \wedge p \wedge q) \vee (r \wedge \neg p \wedge q) \vee (\neg r \wedge \neg p \wedge q))$$

After simplification (applying DeMorgan's law and double negation law), this becomes

$$(\neg r \vee \neg p \vee \neg q) \wedge (\neg r \vee p \vee \neg q) \wedge (r \vee p \vee \neg q)$$

You can check yourself that this formula is CNF and is indeed equivalent to $((p \to q) \wedge q \to p) \wedge (\neg q \vee \neg r)$; that is, it has the same truth table. A DNF for this formula is:

$$(r \wedge p \wedge \neg q) \vee (r \wedge \neg p \wedge \neg q) \vee (\neg r \wedge p \wedge q) \vee (\neg r \wedge p \wedge \neg q) \vee (\neg r \wedge \neg p \wedge \neg q)$$

Note that between the CNF and DNF all 8 truth assignments are mentioned: the falsifying ones in the CNF and satisfying in DNF.

The following puzzle does not relate to the material in this lecture; hopefully we will get to the relevant material next time.

Puzzle 6. Prove that in a big city like Toronto with more than a million people there would be at least two people with the same number of hairs on their heads. Assume that the number of hairs on somebody's head cannot be more than $300,000$.