CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 5

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1.1 Conditional statements

An implication is false in only one case: when it states that T implies F. So taking negation of $p \to q$ gives $p \wedge \neg q$. The easy way to see it is using the definition of implication: $\neg(p \to q) \iff \neg(\neg p \vee q) \iff (\neg\neg p \wedge \neg q) \iff (p \wedge \neg q)$

Example 1. If Jane is in London, then she is in England.
Negation: Jane is in London, and she is not in England (e.g., London, Ontario).

Continuing from the previous lecture. Recall that we looked at the following types of proofs.

1) *Direct proof*: show that if p is true directly.
2) *Proof by contrapositive*: instead of $p \to q$ prove $\neg q \to \neg p$.
3) *Proof by contradiction:* to show that p is true, show that $\neg p \to F$.
4) *Proof by cases:* to show that p is true, prove $(q \to p) \wedge (\neg q \to p)$.

   Lemma 1. *For any natural number n, $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$*

   Here, $\lfloor k \rfloor$ is a *floor* of a (real) number, defined to be the largest integer smaller than or equal to k. For example, $\lfloor 5/2 \rfloor = 2$, and $\lfloor 4/2 \rfloor = 2$. Similarly, a *ceiling* of a number is the smallest integer larger than or equal to that number. So $\lceil 5/2 \rceil = 3$ and $\lceil 4/2 \rceil = 2$. For an integer, both its floor and its ceiling are equal to that integer itself; for a non-integer, the floor is is rounded-down value and the ceiling is rounded up.

   Proof. **Case 1: n is even**. Then $\lfloor (n+1)/2 \rfloor = n/2 = \lceil n/2 \rceil$.
   Case 2: n is odd. Then $\lfloor (n+1)/2 \rfloor = (n+1)/2 = \lceil n/2 \rceil$. ∎

1.2 Valid and invalid arguments

Bertran Russell once said "if 2+2=5, then I am the Pope". His proof was a valid argument (that is, every step logically followed from the previous); but since his premise "2+2=5" was false, it is no wonder that his conclusion was false as well.

If Socrates is a man, then Socrates is mortal
Socrates is a man
$\therefore$ Socrates is mortal

Terminology: the final statement is called *conclusion*, the rest are *premises*. The symbol $\therefore$ reads as *therefore*. An argument is *valid* if no matter what statements are substituted into variables, if all premises are true then the conclusion is true.

The valid form of agument is called *rules of inference*. The most known is called *Modus Ponens* ("method of affirming"). Its contrapositive is called *Modus Tollens* ("method of denying"):

Modus Ponens	*Modus Tollens*
If p then q	If p then q
p	$\neg q$
$\therefore q$	$\therefore \neg p$

This is another way to describe "proof by contrapositive". Similarly we can write the proof by cases, by contradiction, by transitivity and so on. They can be derived from the original logic identities. For example, modus ponens becomes $((p \to q) \wedge p) \to q$.

Example 2. The general form of the proof by cases above can be written as follows:

$p \vee q$	"n is even or n is odd"
$p \to r$	"if n is even then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "
$q \to r$	"if n is odd then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "
$\therefore r$	"therefore $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "

To show that an argument is invalid give a truth table where all premises are true and the conclusion is false. That is, show that a propositional formula of the form $\wedge$(of premises) $\to$ conclusion is not a tautology.

Example 3. For example, although modus ponens is valid, the following is not a valid inference:

$p \rightarrow q$
q INVALID ARGUMENT!
$\therefore p$

We can write it as $((p \rightarrow q) \wedge q) \rightarrow p$ and show that this is not a tautology using a truth table.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$(p \rightarrow q) \wedge q \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	**F**
F	F	T	F	T

The truth table showed us a situation when both premises $(p \rightarrow q)$ and q are true, but the conclusion p is false. Therefore, $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology and thus the argument based on it is not a valid argument.

2 Normal forms of propositional formulas.

Any formula has an equivalent one in a normal form. In computer science, the two normal forms we are interested in are *conjunctive normal form (CNF)* and *disjunctive normal form (DNF)*. The first one is a $\wedge$ of $\vee$s, where $\vee$ is over variables or their negations (*literals*), the second is a $\vee$ of $\wedge$s of literals. A $\vee$ of literals is also called a *clause*, and a $\wedge$ of literals a *term*.

Example 4.

- $\neg p$, x, s are examples of literals, whereas $\neg\neg p$ or $(x \vee y)$ is not a literal
- $(x \vee \neg y \vee z)$, $(\neg p)$ are clauses,
- $(x \wedge \neg y \wedge z)$, $(\neg p)$ are terms.
- $(x \vee \neg z \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg y)$ is a CNF.
- $(x \wedge z) \vee (\neg y \wedge z \wedge x) \vee (\neg x \wedge z)$ is a DNF.
- $(x \wedge \neg(y \vee z) \vee u)$ is neither a CNF nor a DNF, but it is equivalent to the following DNF: $(x \wedge \neg y \wedge \neg z) \vee u$.

Why do we need CNFs and DNFs? Other than the fact that they are convenient normal forms, they are very useful for constricting formulas given a truth table. In particular, DNFs can be constructed from truth table by taking a disjunction (that is, $\vee$) of all satisfying truth assignments and CNFs by taking a conjunctio ($\wedge$) of negations of falsifying truth assignments

Example 5. Recall the truth table in example 3 above. Suppose that you are given just the first two and the last column. How do you construct a formula that would have such a truth table?

Let us start by constructing a DNF formula. It would say, basically, "either the first line satisfies me, or the second, or the last". We write it as $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$. Each term here fully describes one satisfying assignment, and the whole formula is true iff $(p \to q) \wedge q \to p$ is. Now, to construct a formula in CNF we make it say "I am satisfied by neither of the following assignments". One way to write it is as a negation of a DNF formula listing all falsifying assignments. In this example there is just one falsifying assignment, so the formula becomes $\neg(\neg p \wedge q)$. By DeMorgan's law, this is equivalent to $(\neg\neg p \vee \neg q)$, and finally $(p \vee \neg q)$. If there were more than one falsifying assignment, here we would have several of clauses with $\wedge$ between them.

This construction also shows us that for any formula there is an equivalent CNF and an equivalent DNF formula (these are the formulas constructed from the truth table of the original formula). In particular, if the formula is a tautology, its DNF and CNF consist of just $(p \vee \neg p)$ for an arbitrary p, and if it is a contradiction, of $(p \wedge \neg p)$.