CS 2742 (Logic in Computer Science) – Fall 2008
Lecture 32

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10.1 Proving Gödel's incompleteness theorem

Example of a theory of arithmetic: Peano Arithmetic. This was the kind of theory for which Hilbert wanted to prove the consistency.

- Equality is transitive.
- Natural numbers are recursively defined starting from 0 and adding 1 at each step of the recursion. Here, instead of "+1" sometimes people use the "successor" operation: $S(x) = x + 1$.
- For any x, $x + 1 > 0$ (can also be written as $S(x) > 0$).
- Induction works: if $\phi(0)$ holds and $\forall i$, $\phi(i) \to \phi(i + 1)$ then $\forall n, \phi(n)$.
- Addition and multiplication satisfy algebraic axioms: $a+b = b+a$ and so on. Addition is defined using successor as $a + S(b) = S(a + b)$.
- $a \leq b$ is a total order.

A *model* of a theory is a set and interpretations of functions that satisfy the axioms. For example, Euclidean geometry without the parallel lines axiom has different models. E.g., a Mobius strip is not a model of Euclidean geometry, but it is a model of geometry without the fifth postulate.Riemann and Lobachevsky geometries.

A standard model of Peano arithmetic is natural numbers with the usual $=, +, *$ and so on on them.

Theorem 1 (Gödel's incompletenes theorems). *1) Any effectively generated theory which can express elementary arithmetic cannot be both consistent and complete.*

2) Any such theory cannot prove the statement stating its own consistency (unless it is inconsistent and so can prove everything).

Idea of the proof of the first incompleteness theorem: construct a sentence G: "I am not provable". How do you say "I am" in arithmetic? The idea that the formulas can be enumerated (do a countability argument here). This technique is called arithmetization. Here, we end up saying "ϕ is a formula number n", where ϕ says "formula number n is not provable".

Gödel's numbers: a formula is a sequence of symbols $x_1 x_2, x_3 ... x_n$ (for example, view each symbol as a byte). Then $G(x_1 \ldots x_n) = 2^{x_1} \cdot 3^{x_2} \cdot p_n^{x_n}$ where p_n is the n^{th} prime number. Gödel uses this idea both to encode a formula as a number and a sequence of formulas (i.e., a proof) as a number.

Since a relation between formulas and proofs becomes a relation between two numbers, can define a formula $Bew(y) = \exists x(y$ is a Gödel number of a proof of the formula encoded by $y)$. The name Bew comes from German "Beweisbar" which translates as "provable".Now look at $\neg Bew(x)$. Then the statement $p \iff \neg Bew(p)$ says, more or less, "p is a Gödel number of an unprovable formula", and thus p says "my Gödel number is that of an unprovable formula".

For the intuition here, think of the two barbers who shave each other.