

# CS 2742 (Logic in Computer Science) – Fall 2008

## Lecture 3

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### 1.1 Logical equivalences

Recall the puzzle from the previous class: on some island, there are knights (who always tell the truth) and knaves (who always lie). You meet two islanders (call them A and B) and hear the first one say “at least one of us is a knave”. Can you tell whether the islanders are knights or knaves and which islander is which?

We solve this puzzle using a truth table. Take  $p$  :”A is a knight” and  $q$  : “B is a knight.”. Then the sentence “At least one of us is a knave” is translated as  $(\neg p \vee \neg q)$ , since being a knave is the negation of being a knight. Now, we want to know when the truth value of this sentence  $(\neg p \vee \neg q)$  is the same of the truth value of  $p$ : that is, if A is a knight, then what he said must be true, and if A is a knave, then what he said must be false. Let us introduce the notation  $\iff$  to mean *logical equivalence* (that is, two formulas having the same truth values for any truth assignment to their variables). Then the last condition becomes  $p \iff (\neg p \vee \neg q)$ .

We represent this as a truth table as follows:

$p$	$q$	$(\neg p \vee \neg q)$	$p \iff (\neg p \vee \neg q)$
T	T	F	F
T	F	T	T
F	T	T	F
F	F	T	F

As you can see, the only scenario when what A says corresponds correctly with A’s being a knight/knave is the second line: that is, when A is a knight and B is a knave.

**Definition 1** We say that two formulas are logically equivalent ( $A \iff B$ ) if they have the same truth value under any truth assignment.

That is, in a truth table the columns of logically equivalent formulas are identical. This is a semantic notion of equivalence. It can also be defined syntactically:  $p \leftrightarrow q$  defined as  $(p \rightarrow q) \wedge (q \rightarrow p)$ . You can check that this formula holds if and only if  $p$  and  $q$  have the same value.

A useful function of logically equivalent formulas is that if in any formula you replace a subformula by another that is logically equivalent to it, then the value of the whole formula would not change. For example,  $p \wedge (q \vee \neg q)$  is logically equivalent to  $p$  (can check this using a truth table). So if in a formula there is an occurrence of  $p \wedge (q \vee \neg q)$  it can be safely replaced with just  $p$ , thus simplifying the formula.

## 1.2 Logical identities

Now that we have a notion of logical equivalences we can talk about a few identities in propositional logic. We will list them as pairs of equivalent formulas.

Name	$\wedge$ -version	$\vee$ -version
Double negation	$\neg\neg p \iff p$	
DeMorgan's laws	$\neg(p \wedge q) \iff (\neg p \vee \neg q)$	$\neg(p \vee q) \iff (\neg p \wedge \neg q)$
Commutativity	$(p \wedge q) \iff (q \wedge p)$	$(p \vee q) \iff (q \vee p)$
Associativity	$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$	$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$
Distributivity	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$ $p \wedge F \iff F$	$p \vee F \iff p$ $p \vee T \iff T$
Idempotence	$p \wedge p \iff p$	$p \vee p \iff p$
Absorption	$p \wedge (p \vee q) \iff p$	$p \vee (p \wedge q) \iff p$

Notice again (as working our way to boolean algebras) that many of these identities behave just like algebraic and arithmetic identities, with  $\wedge$  behaving like  $\times$ ,  $\vee$  like  $+$ ,  $T$  like 1 and  $F$  like 0. For example, commutativity and associativity laws are the same as for numbers:  $(3 + 2) + 5 = 3 + (2 + 5)$ . One notable exception is that with numbers there is just one form of distributed law, namely the  $\wedge$  form ( $a \times (b + c) = (a \times b) + (a \times c)$ ), and the  $\vee$  form does not hold ( $a + b \times c \neq (a + b) + (a + c)$ ), whereas in logic both forms are true.

## 1.3 Simplifying propositional formulas.

Now we can apply these identities to simplify propositional formulas.

**Example 1**

$$\begin{aligned} & (p \wedge q) \vee \neg(\neg p \vee \neg q) \\ \iff & (p \wedge q) \vee (\neg\neg p \wedge \neg\neg q) && \text{Apply DeMorgan's} \\ \iff & (p \wedge q) \vee (p \wedge q) && \text{Double Negation (twice)} \\ \iff & (p \wedge q) && \text{Idempotence} \end{aligned}$$

**Puzzle 3** What is the value of  $2 + 2 = 4$ ?