

CS 2742 (Logic in Computer Science) – Fall 2008

Lecture 10

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5 Predicate logic

Sometimes we encounter sentences that only have a truth value depending on some parameter. For example, $Even(x)$ which states that the number x is even can be true or false depending on the actual value of x . That is, $Even(5)$ is false, and $Even(10)$ is true.

It is convenient to think of predicates as propositions with parameters. Here, parameters can be numbers, items, etc and there can be infinitely many possibilities for a parameter value. For example, $x^2 > x$ is a predicate with an argument x , where we think of x as a number. Another predicate $Parent(x, y)$ could state that x is a parent of y . Here, it makes sense to think of x and y as people, or at least living creatures. Truth values of a predicate are defined for a given assignment of variables. For example, if $x = 2$, then $x^2 > x$ is true, and if $x = 0.5$, then $x^2 > x$ is false. We call a set of possible objects from which the values of a predicate can come from a *domain* of a predicate.

Here it is convenient to use some set-theoretic notation. A “set” is just a collection of objects (elements). We write $x \in S$ (pronounced “ x is in S ”) to mean that an element x is in a set S . For example, S can be a set of people who wear hats, or numbers greater than 10, or students in a classroom. If an element is not in a set, we use the notation $x \notin S$. Some special kinds of sets are \mathcal{N} , \mathcal{Z} , \mathcal{Q} , \mathcal{R} , \mathcal{C} , which are the sets of natural numbers, integers, rational, real and complex numbers, respectively. Another special set is \emptyset , which is a set containing no elements.

5.1 Quantifiers

Without fixing the values of arguments of a predicate it is not possible to say if the predicate is true or false. That is, unless we want to say that the predicate is false for all possible values

of its arguments (in the domain of this predicate). Here, we need to pay careful attention to what we mean by all possible values: $x^2 \geq x$ is true for and it is false for some rational and real numbers such as 0.5.

Quantifiers are the notational device that allows us to talk about all possible values of arguments and make sentences with truth values out of predicates.

Definition 1. A formula $\forall x A(x)$, where $A(x)$ is a formula containing predicates, is true (on the domain of predicates) if it is true on every value of x from the domain. Here, \forall is called a universal quantifier, usually pronounced as “for all ...”.

For example, $\forall x x^2 \geq x$ states that for every element from the domain the square of that element is greater than the element itself. This formula now has a truth value, provided we know the domain from which x comes from. If the domain is \mathcal{Z} , then the formula is true, and if the domain is \mathcal{Q} , then it is false. Often the domain is written explicitly: $\forall x \in \mathcal{Z} x^2 \geq x$, which is a shortcut for $\forall x (x \in \mathcal{Z} \rightarrow x^2 \geq x)$.

When we want to say that something is not true everywhere, all we need to do is to give a counterexample. E.g., to show that for \mathcal{Q} it is not true that $\forall x x^2 \geq x$ it is enough to give one value on which $x^2 \geq x$ does not hold such as $x = 0.5$. We denote this with the second type of quantifiers, an *existential* quantifier.

Definition 2. A formula $\exists x A(x)$, where $A(x)$ is a formula containing predicates, is true (on the domain of predicates) if it is true on some value of x from the domain. Here, \exists is called a existential quantifier, usually pronounced as “exists ...”.

When doing boolean operations on formulas containing quantifiers, always remember that universal and existential quantifiers are opposites of each other. So,

$$\neg(\forall x A(x)) \iff \exists x \neg A(x) \qquad \neg(\exists x A(x)) \iff \forall x \neg A(x)$$

Now that we have this notation we can define what kinds of formulas we can construct using this language, the *first-order formulas*.

Definition 3. A predicate is a first-order formula (possibly with free variables). A \wedge, \vee, \neg of a first-order formula is a first-order formula. If a formula $A(x)$ has a free variable (that is, a variable x that occurs in some predicates but does not occur under quantifiers such as $\forall x$ or $\exists x$), then $\forall x A(x)$ and $\exists x A(x)$ are also first-order formulas.

Note that this definition is very similar to the definition of propositional formulas except here there are predicates instead of propositions and there are quantifiers.

5.2 English and quantifiers

In English, the closest word to the universal quantifier is “all” or “every”. The closest word to the existential quantifier is “some” and “exists”. But there is one word that can be used as either a universal or an existential quantifier. That is the word *any*. Often we take it to mean a universal quantifier, as in “take any number greater than 1...” (that is, every number greater than 1 would work). But compare the following two sentences:

“I will be happy if I do well in *every* class”.

“I will be happy if I do well in *any* class”.

Here, the word “any” takes the meaning of an existential quantifier: that is, I’ll be happy if there exists some class in which I do well. Please keep this in mind when doing the translations.

Puzzle 8. The first formulation of the famous liar’s paradox, done by a Cretan philosopher Epimenides, stated “All Cretans are liars”. Is this a paradox?