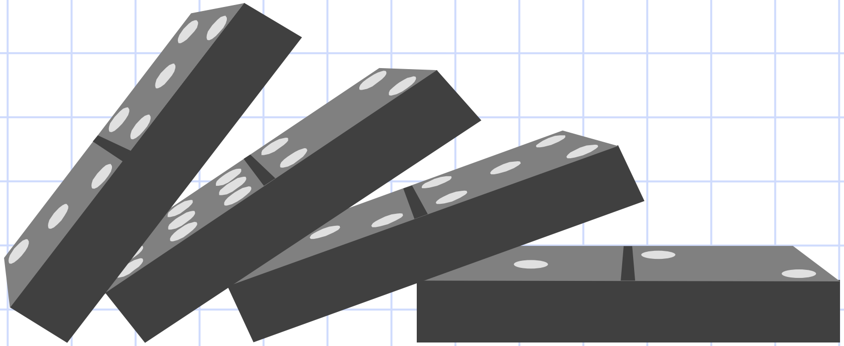
# Induction and loop invariants



Domino Principle: Line up any number of dominos in a row; knock the first one over and they will all fall.

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In algorithm analysis, we are interested in what happens for large input sizes. How do we prove that something is true for an arbitrary large n?

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# Induction: domino principle

In a row of dominos,

- If the first one falls
- If each domino falling knocks the next one
- Then all of them fall.

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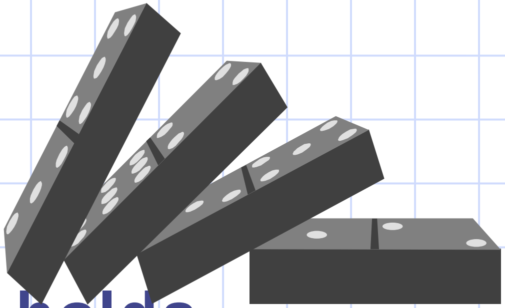
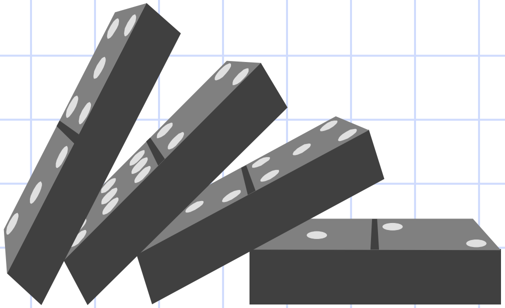
# Induction



- **Statement to prove: for all n, P(n) holds.**
  - For example, for all n, sum of the first n elements is $1+2+\ldots+n=n(n+1)/2$
- **Base Case:**
  - P(0) holds (or P(1) holds).
  - For example, P(1): $1=1(1+1)/2=1$
- **Induction hypothesis:**
  - P(n) holds for some arbitrary n.
  - $1+2+\ldots+n=n(n+1)/2$
- **Induction step.**
  - From the fact that P(n) holds we can derive that P(n+1) holds.

# Induction Step



- **Induction step.**
  - From the fact that $P(n)$ holds (and the base case) we can derive that $P(n+1)$ holds.
  - From $1+2+\ldots n=n(n+1)/2$ derive $1+2+\ldots+n+n+1=(n+1)(n+2)/2$
  - Proof: by induction hypothesis,

$1+\ldots+n+n+1=n(n+1)/2+(n+1)$

- $\qquad\qquad\qquad = (n+1)(n+2)/2$

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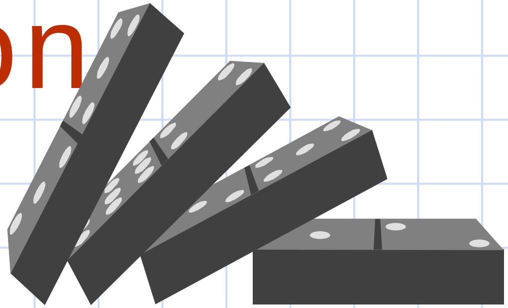
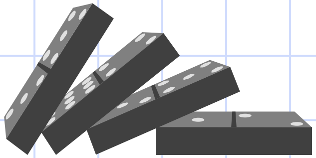
# Structure of induction proof



- **Statement to prove:**
  - For all n > k, P(n) is true.
- **Base Case:**
  - P(k) holds (usually P(0) or P(1)).
- **Induction hypothesis:**
  - P(n) holds for some arbitrary n.
- **Induction step.**
  - Assuming P(n) holds derive that P(n+1) holds.

# Another example



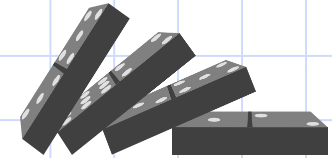
- **Statement to prove:**
  - For all $n>4$, $n^2<2^n$.
- **Base Case:**
  - P(5): $5^2=25 < 2^5=32$
- **Induction hypothesis:**
  - Assume $n^2<2^n$ for $n>4$
- **Induction step.**
  - Assuming $n^2<2^n$ derive $(n+1)^2<2^{(n+1)}$.
  - $(n+1)^2=n^2+2n+1 \le 2n^2 < 2 \cdot 2^n = 2^{(n+1)}$

Because since $n>4$,
$n*n>4n=2n+2n>2n+1$

Induction hypothesis

- **Statement to prove:**
  - For all $n>k$, $P(n)$ is true.
- **Base Case:**
  - $P(k)$ holds.
- **Induction hypothesis:**
  - P(n) holds for some n>k.
- **Induction step.**
  - From P(n) derive P(n+1).

# Strong induction



- **Statement to prove:**
  - For all n>k, P(n) is true.
- **Base Case:**
  - P(k) holds.
- **Induction hypothesis:**
  - P(n) holds for some n>k.
- **Induction step.**
  - From P(n) derive P(n+1).

- **Statement to prove:**
  - Same: for all n>k, P(n).
- **Base Case:**
  - P(k), maybe also P(k+1).
- **Induction hypothesis:**
  - Assume for all m<n P(m) holds.
- **Induction step.**
  - Assuming for all m<n P(m) holds, prove P(n).

Aren't we assuming the same thing as proving?

We are proving P(n) everywhere, assuming it only for the first several elements!

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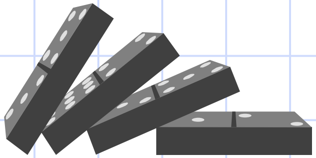
# Strong induction



- **Statement to prove:**
  - For all n, $F(n) < 2^n$
  - F(n) is $n^{th}$ Fibonacci number
- **Base Cases:**
  - $F(0) = 0 < 1$, $F(1) = 1 < 2$
- **Induction hypothesis:**
  - Assume $F(m) < 2^m$ for all $m < n$
- **Induction step.**
  - Assuming induction hypothesis derive $F(n) < 2^n$.
  - $F(n) = F(n-1) + F(n-2)$. By induction hypothesis, $F(n-1) < 2^{n-1}$ and $F(n-2) < 2^{n-2}$.
  - So $F(n) < 2^{n-1} + 2^{n-2} < 2*2^{n-1} = 2^n$

*Strong Induction*

- **Statement to prove:**
  - For all $n > k$, $P(n)$ is true.
- **Base Case:**
  - $P(k)$ holds.
- **Induction hypothesis:**
  - $P(m)$ holds for all m, $k < m < n$,
- **Induction step.**
  - From ind. hyp. derive $P(n)$.

# Algorithm correctness

- ***Precondition*: something that is true before a part of an algorithm (e.g, a loop).**
  - CurrentMax contains A[0].
- ***Postcondition*: something that is true after it finished running.**
  - CurrentMax contains the maximum element of A.
- ***Loop invariant*: .**
  - CurrentMax is the maximum of elements of A seen so far.

**Algorithm** *arrayMax*($A$, $n$)
    **Input** array $A$ of $n$ integers
    **Output** maximum element of $A$

    $currentMax \leftarrow A[0]$
    **for** $i \leftarrow 1$ **to** $n - 1$ **do**
        **if** $A[i] > currentMax$ **then**
            $currentMax \leftarrow A[i]$
    **return** $currentMax$

# Induction and correctness

- ***Precondition:* like base case.**
  - CurrentMax contains A[0].
- ***Loop invariant*: if true at step i, still true after one more loop.**
  - If currentMax contains the maximum of the A[0..i], and the if is executed, currentMax will contain the maximum of A[0..i+1].
- ***Postcondition*: follows from loop invariant.**
  - CurrentMax is the maximum element of A.

**Algorithm** *arrayMax*($A$, $n$)
  **Input** array $A$ of $n$ integer
  **Output** maximum elemen
of $A$

  *currentMax* $\leftarrow A[0]$
  **for** $i \leftarrow 1$ **to** $n - 1$ **do**
    **if** $A[i] >$ *currentMax* **then**
      *currentMax* $\leftarrow$ $A[i]$
  **return** *currentMax*

# Induction and correctness

- ***Precondition: like base case*.**
  - CurrentMax = A[0].
- ***Loop invariant*: if true at step i, still true after one more loop.**
  - Suppose CurrentMax is the max of A[0..i-1]. Two cases:
  - a) A[i]>CurrentMax. Then CurrentMax is the max of A[0..i] since it is in the array (A[i]) and it is the largest (> than previous).
  - b) A[i] is not > than CurrentMax. Then CurrentMax is the max of A[0..i] since it was the max of A[0..i-1] and A[i] is not bigger.
- By induction, true for all iterations including termination.

**Algorithm** *arrayMax*($A$, $n$)
  **Input** array $A$ of $n$ integer
  **Output** maximum elemen
of $A$

  $currentMax \leftarrow A[0]$
  **for** $i \leftarrow 1$ **to** $n - 1$ **do**
    **if** $A[i] > currentMax$
**then**
      $currentMax \leftarrow$
$A[i]$
  **return** $currentMax$

# Algorithm analysis

- **If an algorithm is a sequence of (constantly many) parts, its running time is the maximal over running times of parts.**
  - Eg., if the algorithm sorts inputs in time O(n log n) then looks at each one once in O(n), total time is O(max(n log n, n)) = O(n log n).
- **If an algorithm has k nested loops, with n as a bound, then the running time is $n^k$.**
  - First prefix average algorithm ran in time $n^2$.
- **If an algorithm makes n recursive calls each time calling itself twice, it is exponential. Calling itself once gives linear factor.**
  - First and second Fibonacci algorithms.