Logic: puzzles, truth and human fallacies

COMP2000

Lecture 3

B 5 2 J
Do we think logically?

• You see the following cards. Each has a letter on one side and a number on the other.

• Which cards do you need to turn to check that if a card has a J on it then it has a 5 on the other side?
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B  C  2  5

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  B  5  2  5
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  - B
  - 5
  - 2
  - 5

• Which cards do you need to turn to check that if a card has a J on it then it has a 5 on the other side?
Do we think logically?

• You see the following cards. Each has a letter on one side and a number on the other.

  B 5 J 5

• Which cards do you need to turn to check that if a card has a J on it then it has a 5 on the other side?
Another puzzle

• You are one of the organizers at a mixer; many people are drinking, some are not.
• You need to make sure that nobody underage is drinking: that is, if somebody is drinking, then they are over 19 years old.

• Which category of people will you need to check?
“if ... then” in logic

• Both puzzles have the same structure:

  “if A then B”

• What circumstances make this true?

  – A is true and B is true
  – A is true and B is false
  – A is false and B is true
  – A is false and B is false
A → B

• We make logical conclusions all the time
• But do we always make them “logically”?
• Sometimes people think that “if ... then” goes both ways...
  – If you live in NL, you must pay HST. John lives in BC. Does he pay HST?
  – If today it Tuesday, then there is a COMP2000 lecture. Today is Thursday. Is there a lecture?
Natural vs. Logic language

• Natural languages are ambiguous.
• For example, the word “any” can have different meanings depending on the context:
  • Any = some
    – She will be happy if she can solve any question.
    – She will be happy if she can solve every question.
  • Any = all
    – Any student knows this.
    – Every student knows this.
Language of logic

<table>
<thead>
<tr>
<th>Pronunciation</th>
<th>Notation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>A and B</td>
<td>$A \land B$</td>
<td>True if both A and B are true</td>
</tr>
<tr>
<td>A or B</td>
<td>$A \lor B$</td>
<td>True if either A or B are true (or both)</td>
</tr>
<tr>
<td>If A then B</td>
<td>$A \rightarrow B$</td>
<td>True whenever if A is true, then B is also true</td>
</tr>
<tr>
<td>Not A</td>
<td>$\sim A$</td>
<td>Opposite of A is true, so true if A is false</td>
</tr>
</tbody>
</table>

- Let A be “It is sunny” and B be “it is cold”
  - $A \land B$: It is sunny and cold
  - $A \lor B$: It is either sunny or cold
  - $A \rightarrow B$: If it is sunny, then it is cold
  - $\sim A$: It is not sunny
Language of logic

• Now we can combine these operations to make longer formulas in the language of logic

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<td>A and B</td>
<td>A \B</td>
<td>Both A and B must be true</td>
</tr>
<tr>
<td>A or B</td>
<td>A ∨ B</td>
<td>Either A or B must be true (or both)</td>
</tr>
<tr>
<td>If A then B</td>
<td>A -&gt; B</td>
<td>if A is true, then B is also true</td>
</tr>
<tr>
<td>Not A</td>
<td>~A</td>
<td>Opposite of A is true</td>
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• Let
  • A be “It is sunny”,
  • B be “it is cold”,
  • C be “It’s snowing”
• Let’s make some sentences out of A, B, C
**Language of logic**

- Let
  - A be “It is sunny”, ☀
  - B be “it is cold”, ☃
  - C be “It’s snowing” 🌨

- What are the translations of:
  - $B \land C \rightarrow \lnot A$  
    - IF (cold AND snowing) THEN not sunny
  - $B \rightarrow (C \lor A)$  
    - IF cold THEN (snowing OR sunny)
  - $\lnot A \land A \rightarrow C$  
    - IF (not sunny AND sunny) THEN snowing

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• We talk about a sentence being true or false when the values of the variables are known.

• If we didn’t know whether it is sunny, we would not know whether $A \land B \rightarrow C$ is true or false.

• If a sentence is true for every possible combinations of variable truth assignments, we call it a “tautology”
“if ... then” in logic

• Both puzzles have the same structure:

  “if A then B”

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  – A is true and B is true
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  – A is false and B is true
  – A is false and B is false
Truth tables

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<tr>
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<th>B</th>
<th>not A</th>
<th>A and B</th>
<th>A or B</th>
<th>if A then B</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
<td>False</td>
<td>True</td>
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<td>True</td>
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- Let
  - A be “It is sunny”
  - B be “it is cold”
- It is sunny and cold.
- It is sunny and not cold
- It is not sunny and cold
- It is neither sunny nor cold
Truth tables

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Now, \( \sim A \lor B \) is:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(Not A) or B</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
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Or: the law of excluded middle

• In classical logic, the law of excluded middle says that either a statement or its opposite must be true.
• But here by the opposite we really mean a negation.

- A: It is sunny.
  - ~A: It is not sunny

- A: Today is Tuesday.
  - ~A: Today is not Tuesday

- A: John votes for NDP.
  - ~A: John does not vote for NDP

- A: You are with us
  - ~A: You are not with us.
Negating composite statements

- What is the negation (opposite) of a longer logic statement? Take a truth table column and flip all the values.
  
  – The negation of “A and B”, not(A and B), is true whenever either A or B is false (check the truth table. That is not(A and B) is the same as (not A or not B).
  
  – The negation of “A or B” is true whenever both A and B are false: not(A or B) is the same as (not A and not B).
  
  – Since “if A then B” is the same as “not A or B”, its negation is “A and not B”. Remember that “if A then B” is only false when A is true, and B is false; check that “A and not B” is true in exactly the same scenario.
NOT’ing longer sentences

• For a longer combination, start with the connective applied last when computing a truth table:
  • “not (if (A or not B) then (A and C))” becomes
  • (A or not B) and not(A and C) by negating “if... Then..”
  • (A or not B) and (not A or not C) by negating the last “and”

– Let A be “it’s sunny” and B “it’s cold”.
  • “It’s sunny and cold today”! -- No, it’s not!
  • That could mean
    – No, it’s not sunny.
    – No, it’s not cold.
    – No, it’s neither sunny nor cold.
  • In all of these scenarios, “It’s either not sunny or not cold” is true.
More on “not if.. then”

- Remember that for “if A then B” there is only one scenario when it is false: it is when A is true, and B is false.
- Let A be “it’s raining” and B “it is cloudy”. Then “if A then B” means if it is raining it must be cloudy.
- “not (if A then B)” means “it’s not true that when it is raining it must be cloudy”. Or, equivalently, “it’s raining, and it is not cloudy” (there is probably a rainbow then, too).
- “not (if A then B) is definitely not a negation of “if B then A”! 
Or: elusive, not exclusive.

- I like one of the shapes.
- I like one of the colours.

- I like a figure if it has either my favourite shape or my favourite colour.

- I like the blue triangle. What can you say about the rest?
Proof vs. disproof

• To prove that something is (always) true:
  – Make sure it holds in every case.
    • I have classes on every day that starts with T. I have classes on Tuesday and Thursday (and Friday, but that’s irrelevant).

  – Or assume it does not hold, and then get something strange as a consequence
    • Suppose there are finitely many prime numbers. What divides the number that’s a product of all primes +1?

• To disprove that something is always true:
  – Give just one example where it breaks down.
    • I have classes every day! – No, you don’t have classes on Saturday!
    • All girls hate math! – No, I love math and I am a girl :(
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Knights and knaves

• On a mystical island, there are two kinds of people: knights and knaves.

  • Knights always say the truth.

  • Knaves always lie.
Knights and knaves

• On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

• Puzzle 1: You meet two people on the island, Arnold and Bob. Arnold says “at least one of us is a knave”. Is Arnold a knight or a knave? What about Bob?
Knights and knaves

- On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

- Puzzle 2: You meet two people on the island, Arnold and Bob. Arnold says “Either I am NOT a knight, or Bob is a knave” Is Arnold a knight or a knave? What about Bob?
Knights and knaves

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• Puzzle 3: You see three islanders talking to each other, Arnold, Bob and Charlie. You ask Arnold “Are you a knight?”, but can’t hear what he answered. Bob pitches in: “Arnold said that he is a knave!” and Charlie interjects “Don’t believe Bob, he’s lying”. Out of Bob and Charlie, who is a knight a who is a knave?
Twins puzzle

• There are two identical twin brothers, Dave and Jim.
• One of them always lies; another always tells the truth (like knights and knaves).
• Suppose you see one of them and you want to find out his name.
• How can you learn if you met Dave or Jim by asking just one short yes-no question? You don’t know which one of them is the liar.