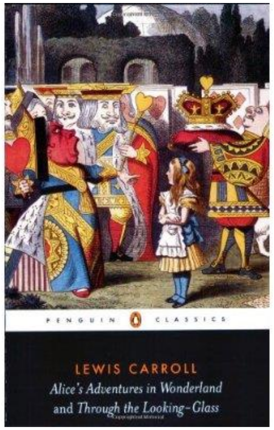
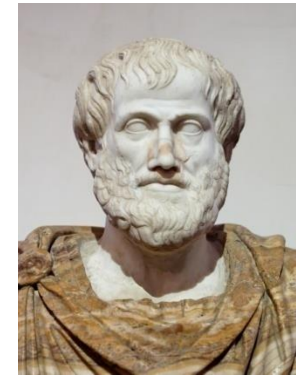




COMP 1002

Intro to Logic for Computer Scientists

Lecture 8

B 5 2 J

Admin stuff

- Labs start next Monday/Wednesday
  - Lab 1 posted
- Assignment 1 posted. Due Jan 24
  - We switched to 6 assignments at 5% each
  - Please type up your assignments!
- Midterm: Feb 28th
  - 3 assignments handed in and marked by the drop date, but not the midterm.

Puzzle 8

- Suppose that nobody in our class carries more than 10 pens.
- There are 108 students in our class.

- Prove that there are at least 2 students in our class who carry the same number of pens.
  – In fact, there are at least 10 who do.

Pigeonhole Principle

- Suppose that nobody in our class carries more than 10 pens.
- There are 108 students in our class.

- Prove that there are at least 2 students in our class who carry the same number of pens.
  - In fact, there are at least 10 who do.

- The Pigeonhole Principle:
  - If there are n pigeons
  - And n-1 pigeonholes
  - Then if every pigeon is in a pigeonhole
  - At least two pigeons sit in the same hole

Pigeonhole Principle

- Suppose that nobody in our class carries more than 10 pens. There are 108 students in our class. Prove that there are at least 2 students in our class who carry the same number of pens.
  - In fact, there are at least 10 who do.
- The Pigeonhole Principle:
  - If there are n pigeons and n-1 pigeonholes
  - Then if every pigeon is in a pigeonhole
  - At least two pigeons sit in the same hole

- Applying to our problem:
  - n-1 = 11 possible numbers of pens (from 0 to 10)
  - Even with n=12 people, there would be 2 who have the same number.
  - If there were less than 10, say 9 for each scenario, total would be 101.
  - Note that it does not tell us which number or who these people are!

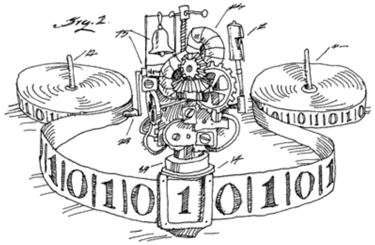
Pigeonhole Principle

- Prove that at least two people in our class of 108 will get the same mark in the class (0 to 100)

- The Pigeonhole Principle:
  - If there are more pigeons than holes
    - Eg n pigeons and at most n-1 holes
  - Then if every pigeon is in a pigeonhole
  - At least two pigeons sit in the same hole

- Applying to our problem:
  - There are n=108 people in our class.
  - There are 101 < n-1=107 possible marks.
  - By the Pigeonhole Principle, at least two people get the same mark

Automated provers



- How to make an automated prover which checks whether a formula is a tautology?
  - And so can check if an argument is valid, etc.
- Truth tables:
  - easy to program, but proofs are huge.
- Natural deduction:
  - proofs might be smaller than a truth table
    - Are they always? Good question...
  - even if there is a small proof, how can we find one quickly?
    - Nobody knows...

Resolution proofs

- Middle ground: use the **resolution rule**:
  - Basis for many practical provers (SAT solvers).
  - Used in verification, scheduling, etc...

$$\begin{array}{l} C \lor x \\ D \lor \neg x \\ \hline \therefore C \lor D \end{array} \qquad \begin{array}{l} y \lor \neg z \lor w \\ u \lor \neg w \\ \hline \therefore y \lor \neg z \lor u \end{array} \qquad \begin{array}{l} y \lor w \lor \neg z \\ \neg z \lor \neg w \\ \hline \therefore y \lor \neg z \end{array}$$

- Ignore order in an OR and remove duplicates.

Resolution proofs

- Rather than proving that F is a tautology, prove that $\neg F \equiv FALSE$. That is, a proof of F is a **refutation** of $\neg F$
  - To check that an argument is valid, refute AND of premises AND NOT conclusion.

- Last step of the resolution refutation of $\neg F$:
  - from x and $\neg x$ derive FALSE, for some variable x.
  - If you cannot derive anything new, then the formula is satisfiable.

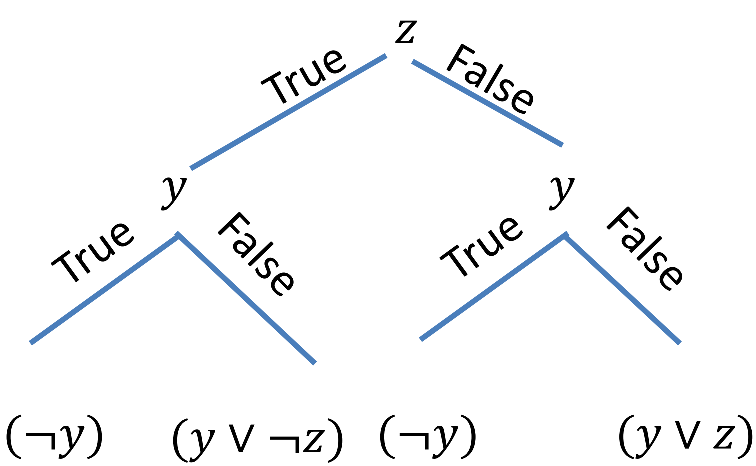
$$(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$$

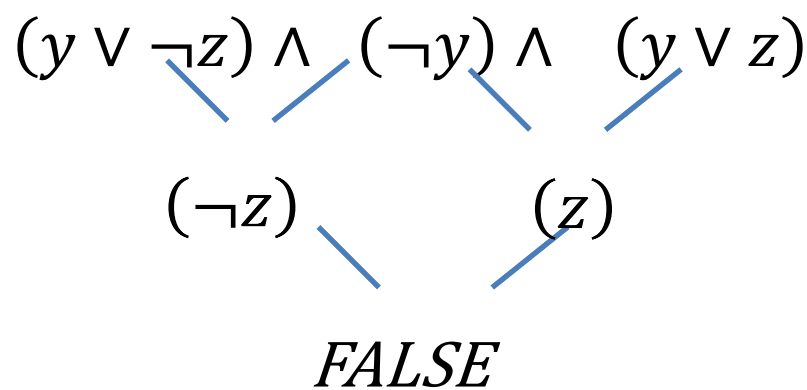
$$(\neg z) \qquad (z)$$

$$FALSE$$

Decision trees

- Resolution rule:
  - from clauses $(C \vee x)$ and $(D \ \vee \neg x)$
    - Where C and D are ORs of possibly negated variables.
  - derive clause $(C \vee D)$
  - Show that all clauses cannot be satisfied at the same time.
- An "upside-down" view: decision tree.
  - For every assignment, some clause is false.





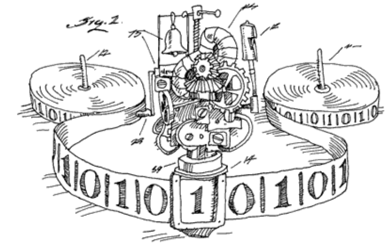
CNF (Product of Sums)

- Resolution works best when the formula is of the special form: it is an AND of ORs of (possibly negated) variables (called **literals**).

- This form is called a **Conjunctive Normal Form**, or **CNF**.
  - Also known as Product of Sums
  - $(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$ is a CNF
  - $(x \vee \neg y \vee z)$ is a CNF. So is $(x \wedge \neg y \wedge z)$.
  - $(x \vee \neg y \wedge z)$ is not a CNF

- An AND of CNF formulas is a CNF formula.
  - So if all premises are CNF and the negation of the conclusion is a CNF, then AND of premises AND NOT conclusion is a CNF.

CNF



- To convert a formula into a CNF.
  - Open up the implications to get ORs.
  - Get rid of double negations.
  - Convert $F \vee (G \wedge H)$ to $(F \vee G) \wedge (F \vee H)$.

- Example: $A \rightarrow B \wedge C$

  $\equiv \neg A \vee B \wedge C$

  $\equiv (\neg A \vee B) \wedge (\neg A \vee C)$

- In general, CNF can become quite big, especially when have $\leftrightarrow$. There are tricks to avoid that...

Natural deduction

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake
- Therefore, the treasure is not in the kitchen.

1. If A then not B
2. If C then B
3. A
4. C or D
5. If E then F
6. Not B
7. Not C
8. D

Treasure hunt

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the font yard is an elm, then the treasure is in the kitchen.
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is buried under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

1. $A \rightarrow \neg B$
2. $C \rightarrow B$
3. A
4. C ∨ D
5. E → F

1. $\neg A \vee \neg B$
2. $\neg C \vee B$
3. A
4. C ∨ D
5. $\neg E \vee$ F

Conclusion: D

Check validity of the argument using resolution

$$(\neg A \vee \neg B) \wedge (\neg C \vee B) \wedge (A) \wedge (C \vee D) \wedge (\neg E \vee F) \wedge \neg D$$

$(\neg A \vee \neg C)$

(C)

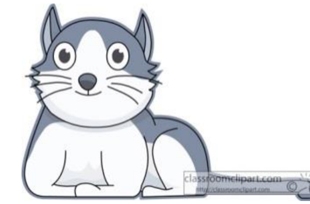
$(\neg C)$

$FALSE$

Resolution and Pigeons

- It is not that hard to write the Pigeonhole Principle as a tautology
- But we can prove that resolution has trouble with this kind of reasoning
  - the smallest resolution proof of this tautology is exponential size!
- By contrast, natural deduction (and you!) can figure it out fairly quickly
  - though it is not straightforward.
- The problem is that resolution cannot count.
  - But ability to count makes things harder...


Meow-stery

- One evening there was a cat fight in a family consisting of a mother cat, a father cat, and their son and daughter kittens.
- One of these four cats attacked and bit another!

- One of the cats watched the fight.
- The other one hissed at the fighters.

- These are the things we know for sure:
  - 1. The watcher and the hisser were not of the same sex.
  - 2. The oldest cat and the watcher were not of the same sex.
  - 3. The youngest cat and the victim were not of the same sex.
  - 4. The hissing cat was older than the victim.
  - 5. The father was the oldest of the four.
  - 6. The attacker was not the youngest of the four.

- Which nasty cat was the attacker?