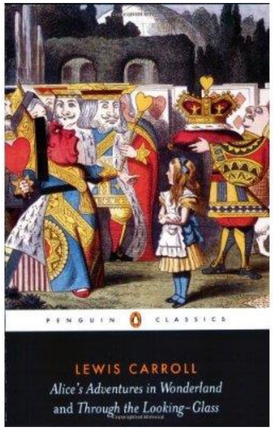
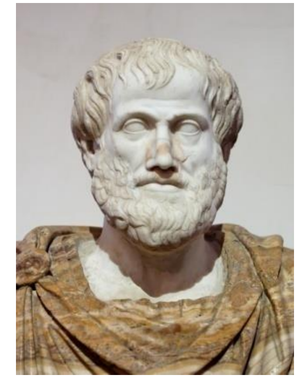




COMP 1002

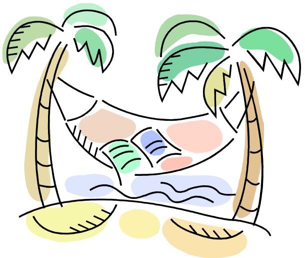
Intro to Logic for Computer Scientists

Lecture 3

Admin stuff

- Make-up lecture next Monday, Jan 14
- 10am to 12pm in C 3033

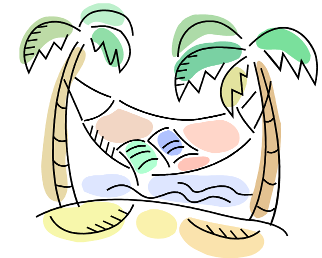

Knights and knaves



- On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

- Puzzle 1: You meet two people on the island, Arnold and Bob. Arnold says “Either I am a knave, or Bob is a knight”. Is Arnold a knight or a knave? What about Bob?


Knights and knaves



- Puzzle 1: You meet two people on the island, Arnold and Bob. Arnold says "Either I am a knave, or Bob is a knight". Is Arnold a knight or a knave? What about Bob?
  - A: Arnold is a knight
  - B: Bob is a knight
  - Formula: $\neg A \vee B$: "Either Arnold is a knave, or Bob is a knight"
  - Want: scenarios where either both A is a knight and the formula is true, or A is a knave and the formula is false. Use "if and only if" notation: $(\neg A \vee B) \leftrightarrow A$. True if both formulas have same value

A	B	$\neg A$	$\neg A \vee B$	$(\neg A \vee B) \leftrightarrow A$
True	*True*	False	True	True
True	*False*	False	False	False
False	*True*	True	True	False
False	*False*	True	True	False

Special types of sentences

- A sentence that has a satisfying assignment is **satisfiable.**
  - Some row in the truth table ends with *True*.
  - Example: $B \rightarrow A$

A	B	$B \rightarrow A$
True	*True*	*True*
True	*False*	*True*
False	*True*	*False*
False	*False*	*True*

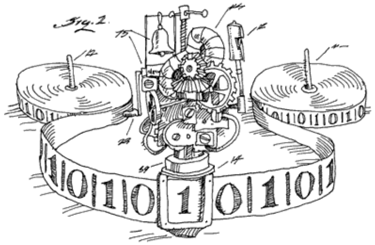
- Sentence is a **contradiction:**
  - All assignments are falsifying.
  - All rows end with *False.*
  - Example: $A \land \neg A$

A	$A \land \neg A$
True	*False*
False	*False*

- Sentence is a **tautology:**
  - All assignments are satisfying
  - All rows end with *True*.
  - Example: $B \rightarrow A \lor B$

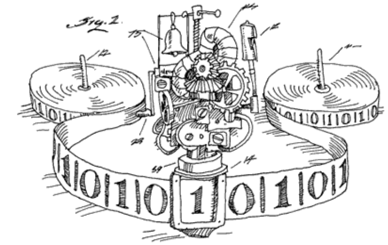
A	B	$A \lor B$	$B \rightarrow A \lor B$
True	*True*	*True*	*True*
True	*False*	*True*	*True*
False	*True*	*True*	*True*
False	*False*	*False*	*True*

Determining formula type



- How long does it take to check if a formula is satisfiable?
  - If somebody gives you a satisfying assignment, then in time roughly the size of the formula.
    - On a m-symbol formula, take time O(m) = constant * m, for some constant depending on the computer/software.
  - What if you don't know a satisfying assignment? How hard it is to find it?
    - Using a truth table: in time $O(m * 2^n)$ on a length m n-variable formula.
    - Is it efficient?...

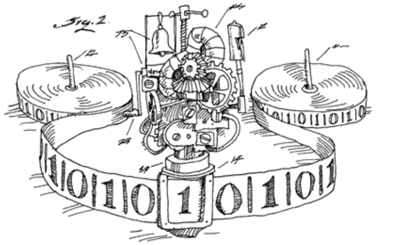
Complexity of computation



- Would you still consider a problem really solvable if it takes very long time?
  - Say 10^n steps on an n-symbol string?
  - At a billion (10^9) steps per second (~1GHz)?
  - To process a string of length 100…
  - will take $10^{100}/10^9$ seconds, or ~3×10^{72} centuries.

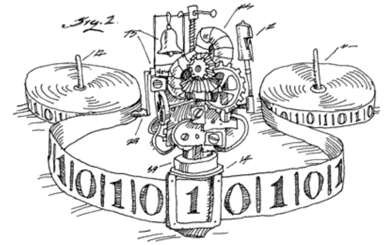
  - Age of the universe: about 1.38×10^{10} years.
  - Atoms in the observable universe: 10^{78}-10^{82}.

Complexity of computation



- What strings do we work with in real life?
  - A DNA string has 3.2×10^9 base pairs
  - A secure key in crypto: 128-256 bits
  - Number of Walmart transactions per day: 10^6.
  - URLs searched by Google in 2012: $3 \text{x} 10^{12}$.

Determining formula type



- How long does it take to check if a formula is satisfiable?
  - Using a truth table: in time $O(m * 2^n)$ on a length m n-variable formula.

  - Is it efficient?
    - Not really!
    - Formula with 100 variables is already too big!
    - In software verification: millions of variables!
  - Can we do better?

A million-dollar question!

Logical equivalence

- Two formulas F and G are **logically equivalent** ($F \Leftrightarrow G$ or $F \equiv G$) if they have the same value for every row in the truth table on their variables.
  - $A \wedge \neg A \equiv False$ (same as saying it is a contradiction)
  - $(\neg A \vee B) \equiv (A \rightarrow B)$
  - $(A \leftrightarrow B) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$
    - $\leftrightarrow$ is sometimes called the "biconditional"
    - $\leftrightarrow$ often pronounced as "if and only if", or "iff"

- Useful fact: proving that $F \equiv G$ can be done by proving that $F \leftrightarrow G$ is a tautology

Double negation

- Negation cancels negation
  - $\neg\neg A \equiv A$
  - “I do not disagree with you” = “I agree with you”

- For a human brain, harder to parse a sentence with multiple negations:
  - Alice says: “I refuse to vote against repealing the ban on smoking in public. “
    - Does Alice like smoking in public or hate it?

De Morgan's Laws

- Simplifying negated formulas
  - For AND: $\neg\,(A \land B)$ is equivalent to $(\neg A \lor \neg B)$
  - For OR: $\neg\,(A \lor B) \equiv (\neg A \land \neg B)$

- Example:
  - $\neg\,(\neg A \lor B)$ is $\neg\neg\,A \land \neg B$, same as $A \land \neg B$
  - So, since $(A \to B)$ is equivalent to $(\neg A \lor B)$,

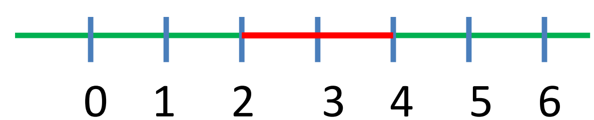
    $\neg(A \to B)$ is equivalent to $A \land \neg B$

De Morgan’s laws: examples

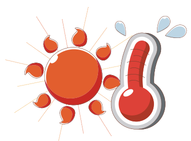
– Let A be “it’s sunny” and B “it’s cold”.
  - “It’s sunny and cold today”! -- No, it’s not!
  - That could mean
    - – No, it’s not sunny.
    - – No, it’s not cold.
    - – No, it’s neither sunny nor cold.

  - In all of these scenarios, “It’s either not sunny or not cold” is true.

– Let A be “x < 2”, B be “x > 4”.



  - “Either $x < 2$ or $x > 4$“ – No, it is not!
  - Then $2 \leq x \leq 4$



More examples

- Let A be "I play" and B "I win".
  - $A \to B$: "If I play, then I win"
  - Equivalent to $\neg A \vee B$: "Either I do not play, or I win".

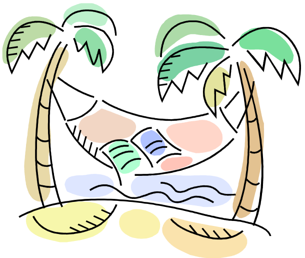
- Negation: $\neg(A \to B)$: "It is not so that if I play then I win".
  - By de Morgan's law: $\neg(\neg A \vee B) \equiv (\neg\neg A \wedge \neg B)$
  - By double negation: $(\neg\neg A \wedge \neg B) \equiv (A \wedge \neg B)$
  - So negation of "If I play then I win" is "I play and I don't win".

Longer example of negation

- Start with the outermost connective and keep applying de Morgan's laws and double negation.
- Stop when all negations are on variables.

- $\neg\,(\,(A \vee \neg B) \rightarrow \ (\neg A \wedge C))$
  - $(A \vee \neg B) \wedge \neg(\neg A \wedge C)$ (negating $\rightarrow$)
  - $(A \vee \neg B) \wedge (\neg\neg A \vee \neg C)$ (de Morgan)
  - $(A \vee \neg B) \wedge (A \vee \neg C)$ (removing $\neg\neg$)


Knights and knaves



- On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.
- Puzzle 2: You see three islanders talking to each other, Arnold, Bob and Charlie.
  - You ask Arnold "Are you a knight?", but can't hear what he answered.
  - Bob pitches in: "Arnold said that he is a knave!"
  - and Charlie interjects "Don't believe Bob, he's lying".
  - Out of Bob and Charlie, who is a knight/knave?