Admin stuff

• Make-up lecture next Monday, Jan 14
• 10am to 12pm in C 3033
Knights and knaves

• On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

• Puzzle 1: You meet two people on the island, Arnold and Bob. Arnold says “Either I am a knave, or Bob is a knight”. Is Arnold a knight or a knave? What about Bob?
Knights and knaves

• Puzzle 1: You meet two people on the island, Arnold and Bob. Arnold says “Either I am a knave, or Bob is a knight”. Is Arnold a knight or a knave? What about Bob?
  – A: Arnold is a knight
  – B: Bob is a knight
  – Formula: \( \neg A \lor B \): “Either Arnold is a knave, or Bob is a knight”
  – Want: scenarios where either both A is a knight and the formula is true, or A is a knave and the formula is false. Use “if and only if” notation: \((\neg A \lor B) \leftrightarrow A\). True if both formulas have same value

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(\neg A)</th>
<th>(\neg A \lor B)</th>
<th>((\neg A \lor B) \leftrightarrow A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
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Special types of sentences

• A sentence that has a satisfying assignment is **satisfiable**.
  – Some row in the truth table ends with **True**.
  – Example: \( B \rightarrow A \)

• Sentence is a **contradiction**:
  – All assignments are falsifying.
  – All rows end with **False**.
  – Example: \( A \land \neg A \)

• Sentence is a **tautology**:
  – All assignments are satisfying
  – All rows end with **True**.
  – Example: \( B \rightarrow A \lor B \)
Determining formula type

• How long does it take to check if a formula is satisfiable?
  – If somebody gives you a satisfying assignment, then in time roughly the size of the formula.
    • On a m-symbol formula, take time $O(m) = \text{constant} \times m$, for some constant depending on the computer/software.
  – What if you don’t know a satisfying assignment? How hard it is to find it?
    • Using a truth table: in time $O(m \times 2^n)$ on a length m n-variable formula.
    • Is it efficient?...
Complexity of computation

• Would you still consider a problem really solvable if it takes very long time?
  – Say $10^n$ steps on an n-symbol string?
  – At a billion ($10^9$) steps per second (~1GHz)?
  – To process a string of length 100...
  – will take $10^{100}/10^9$ seconds, or $\sim 3 \times 10^{72}$ centuries.

  – Age of the universe: about $1.38 \times 10^{10}$ years.
  – Atoms in the observable universe: $10^{78}-10^{82}$. 
Complexity of computation

- What strings do we work with in real life?
  - A DNA string has $3.2 \times 10^9$ base pairs
  - A secure key in crypto: 128-256 bits
  - Number of Walmart transactions per day: $10^6$.
  - URLs searched by Google in 2012: $3 \times 10^{12}$. 
Determining formula type

• How long does it take to check if a formula is satisfiable?
  – Using a truth table: in time $O(m \times 2^n)$ on a length $m$ $n$-variable formula.
  – Is it efficient?
    • Not really!
    • Formula with 100 variables is already too big!
    • In software verification: millions of variables!
  – Can we do better?

A million-dollar question!
Logical equivalence

• Two formulas F and G are logically equivalent \((F \iff G)\) if they have the same value for every row in the truth table on their variables.
  
  – \(A \land \neg A \equiv False\) (same as saying it is a contradiction)
  
  – \((\neg A \lor B) \equiv (A \rightarrow B)\)
  
  – \((A \leftrightarrow B) \equiv (A \rightarrow B) \land (B \rightarrow A)\)
    
    • \(\iff\) is sometimes called the “biconditional”
    
    • \(\iff\) often pronounced as “if and only if”, or “iff”

• Useful fact: proving that \(F \equiv G\) can be done by proving that \(F \leftrightarrow G\) is a tautology
Double negation

• Negation cancels negation
  – \( \neg \neg A \equiv A \)
  – “I do not disagree with you” = “I agree with you”

• For a human brain, harder to parse a sentence with multiple negations:
  – Alice says: “I refuse to vote against repealing the ban on smoking in public. “
    • Does Alice like smoking in public or hate it?
De Morgan’s Laws

• Simplifying negated formulas
  – For AND: \( \neg (A \land B) \) is equivalent to \( (\neg A \lor \neg B) \)
  – For OR: \( \neg (A \lor B) \equiv (\neg A \land \neg B) \)

• Example:
  – \( \neg (\neg A \lor B) \) is \( \neg \neg A \land \neg B \), same as \( A \land \neg B \)
  – So, since \( (A \rightarrow B) \) is equivalent to \( (\neg A \lor B) \),
    \( \neg (A \rightarrow B) \) is equivalent to \( A \land \neg B \)
De Morgan’s laws: examples

— Let A be “it’s sunny” and B “it’s cold”.
  • “It’s sunny and cold today”! — No, it’s not!
  • That could mean
    — No, it’s not sunny.
    — No, it’s not cold.
    — No, it’s neither sunny nor cold.
  • In all of these scenarios, “It’s either not sunny or not cold” is true.

— Let A be “$x < 2$”, B be “$x > 4$”.
  • “Either $x < 2$ or $x > 4$” — No, it is not!
  • Then $2 \leq x \leq 4$
More examples

– Let $A$ be “I play” and $B$ “I win”.
  
  • $A \rightarrow B$: “If I play, then I win”
  
  • Equivalent to $\neg A \lor B$: “Either I do not play, or I win”.

– Negation: $\neg(A \rightarrow B)$: “It is not so that if I play then I win”.
  
  • By de Morgan’s law: $\neg(\neg A \lor B) \equiv (\neg\neg A \land \neg B)$
  
  • By double negation: $(\neg\neg A \land \neg B) \equiv (A \land \neg B)$
  
  • So negation of “If I play then I win” is “I play and I don’t win”.
Longer example of negation

• Start with the outermost connective and keep applying de Morgan’s laws and double negation.
• Stop when all negations are on variables.

• \( \neg ( (A \lor \neg B) \rightarrow (\neg A \land C) ) \)
  • \( (A \lor \neg B) \land \neg(\neg A \land C) \) (negating \( \rightarrow \))
  • \( (A \lor \neg B) \land (\neg \neg A \lor \neg C) \) (de Morgan)
  • \( (A \lor \neg B) \land (A \lor \neg C) \) (removing \( \neg \neg \))
Knights and knaves

• On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

• Puzzle 2: You see three islanders talking to each other, Arnold, Bob and Charlie.
  – You ask Arnold “Are you a knight?”, but can’t hear what he answered.
  – Bob pitches in: “Arnold said that he is a knave!”
  – and Charlie interjects “Don’t believe Bob, he’s lying”.
  – Out of Bob and Charlie, who is a knight/knave?