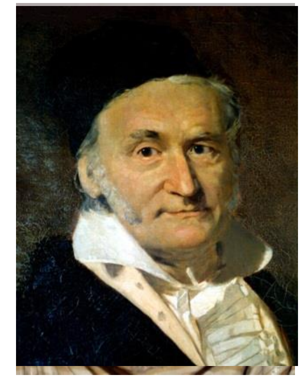

COMP 1002

Logic for Computer Scientists

Lecture 28

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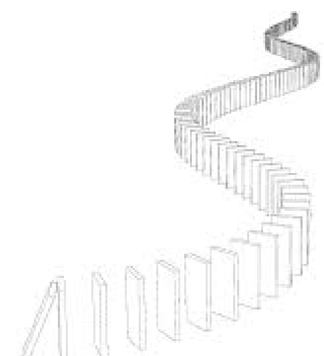
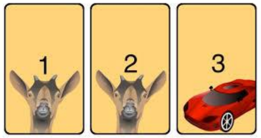
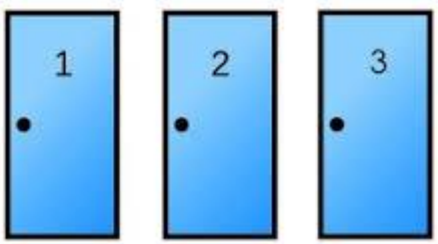


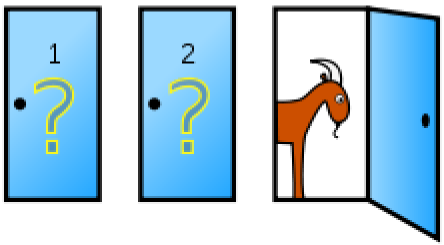


Puzzle: Monty Hall problem

- Let's make a deal!
  - A player picks a door.
  - Behind one door is a car.
  - Behind two others are goats.



- A player chooses a door.
  - A host opens another door
  - Shows a goat behind it.
  - And asks the player if she wants to change her choice.



- Should she switch?

Probabilities of events

- Probability of an event A is a sum of probabilities of the outcomes in A:
  – Pr(A)=$\Sigma_{\{a \in A\}} \Pr(a)$

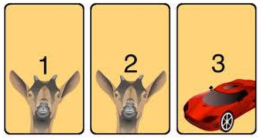
- Probability of A not occurring:
  – $\Pr(\bar{A}) = 1 - \Pr(A)$

- Probability of the union of two events (either A or B happens) is

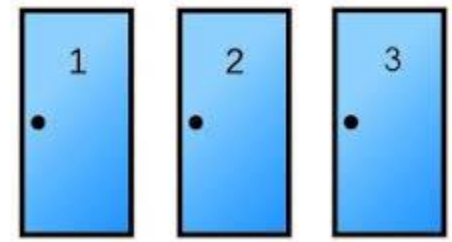
$$\Pr(A \cup \mathrm{B}) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

    - By principle of inclusion-exclusion
  – If A and B are disjoint, then $\Pr(A \cap B) = 0$, so $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

- In general, if events $A_1 \ldots A_n$ are pairwise disjoint
    - that is, $\forall i, j$ if $i \neq j$ then $A_i \cap A_j = \emptyset$
  – Then Pr($\cup_{i=1}^{n} A_i) = \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \Sigma_{i=1}^{n} \Pr(A_i)$
    - That is, probability of that any of the events happens is the sum of their individual probabilities.



Puzzle: Monty Hall problem



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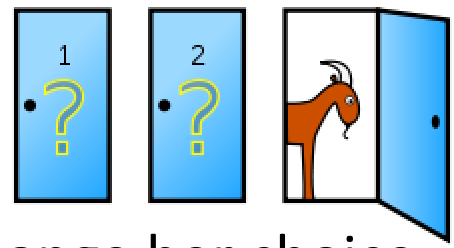


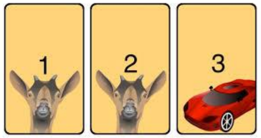
- A player chooses a door.
  - A host opens another door
  - Shows a goat behind it.
  - And asks the player if she wants to change her choice.
- Should she switch?
  - Originally, probability of picking the car is 1/3
  - If she first picked a door with a car: (1/3 probability)
    - Then she would switch to a goat.
  - If she first picked a door with a goat (2/3 probability)
    - Then she would switch to a car.

Conditional probabilities

- **Conditional probability** of an event A given event B, denoted $\Pr(A|B)$, is the probability of A if we know that B occurred.
  - Probability of car a behind door 2 if we chose door 1, and door 3 had a goat behind it.
- So it is probability of both A and B, given that we know B happened for sure:

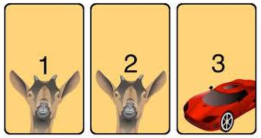
$$\Pr(A|B) = \frac{\Pr(A \cap B\ )}{\Pr(B)}$$

  - Assume that $\Pr(B) > \ 0$: after all, B did happen.



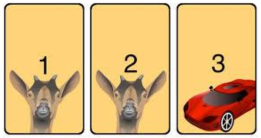
Independent events

- If knowing B gives us no information about A and vice versa, then A and B are **independent** events:
  - Then Pr(A) = Pr(A|B) = $\frac{\Pr(A \cap B)}{\Pr(B)}$.
  - So A and B are independent iff $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$.
    - In general, events $A_1 \ldots A_n$ can be **pairwise independent** (that is, any two A_i, A_j) are independent, or (stronger condition) **mutually independent**: $\Pr(\cap_{i=1}^{n} A_i) = \Pi_{i=1}^{n} \Pr(A_i)$
  - For example, different coin tosses/dice rolls are usually considered independent.



Bayes theorem

- Consider a medical test that
  - Has false positive rate of 3% (healthy labeled as sick)
    - Specificity 97%
  - Has false negative rate of 1% (sick labeled as healthy).
    - Sensitivity 99%
- What is the probability that a person has the disease given that the test came positive?
  - Let A: person tested positive, B: person is sick. $\Pr(B|A)$?
  - Pr(A|B) = 0.99, $\Pr(\bar{A}|B) = 0.01\ldots$
- Not enough information!



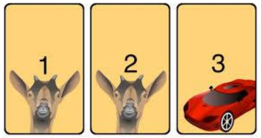
Bayes theorem

- **Bayes theorem** allows us to get $\Pr(B|A)$ from $\Pr(A|B)$, if we know probabilities of A and B:

$$\Pr(B|A) = \frac{\Pr(A|B)\cdot\Pr(B)}{\Pr(A)} = \frac{\Pr(A|B)\cdot\Pr(B)}{\Pr(A|B)\Pr(B)+\Pr(A|\bar{B})\Pr(\bar{B})}$$

- Proof:
  - $\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$. $\Pr(B|A) = \frac{\Pr(A\cap B)}{\Pr(A)}$.
  - $\Pr(\mathrm{A} \cap B) = \Pr(A|B)\Pr(B) = \Pr(B|A)\Pr(A)$
  - So $\Pr(B|A) = \Pr(A|B)\Pr(B)\,/\Pr(A)$

- The formula $\Pr(\mathrm{A}) = \Pr(A|B)\Pr(B) + \Pr(A|\bar{B})\Pr(\bar{B})$ comes from writing probability of A (e.g., a positive test) as sum of probabilities of A for B (sick people) and for $\bar{B}$ (healthy people).



$$\Pr(B|A) = \frac{\Pr(A|B)\cdot\Pr(B)}{\Pr(A)}$$

Bayes theorem

- Consider a medical test that
  - Has false positive rate of 3% (healthy labeled as sick).
  - Has false negative rate of 1% (sick labeled as healthy).
  - Tests for a disease that occurs in 5 in 1000 people.
- What is the probability that a person has the disease given that the test came positive?
  - Let A: person tested positive, B: person is sick. $\Pr(B|A)$?
  - Pr(A|B) = 0.99, $\Pr(\bar{A}|B) = 0.01$. Pr($\bar{A}|\bar{B}$) = 0.97 , $\Pr(A|\bar{B}) = 0.03$
  - $\Pr(B) = 0.005$.
  - $\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|\bar{B})\Pr(\bar{B})$ = 0.0348
  - By Bayes theorem, Pr(B|A) = $\frac{\Pr(A|B)\,\Pr(B)}{\Pr(A)}$ = 0.1422
- So the probability that a person who tested positive has the disease is just 0.1422, that is, 14.22%.
  - By a similar argument, probability that a person who tested negative does not have a disease is whopping 0.99995 = 99.995%.

Hat-check problem

- Suppose n men came to an event, and checked in their hats at the door.
  - On the way out, in a hurry, they each picked up a random hat.
  - On average, how many men picked their own hat?

Expectations

- Often we are interested in what outcome we would see "on average".
  - How fast does this program run "on average"?

- Suppose that possible outcomes of an experiment are numbers $a_1, \dots, a_n$
  - E.g., time a program takes to sort n elements

- Its **expected value (mean)** is $\Sigma_{k=1}^{n} a_k \mathrm{Pr}(a_k)$
  - Often phrased in terms of a "random variable" X, where X is a *function* from outcomes to numbers.
  - Write E(X) to mean the expectation of X.