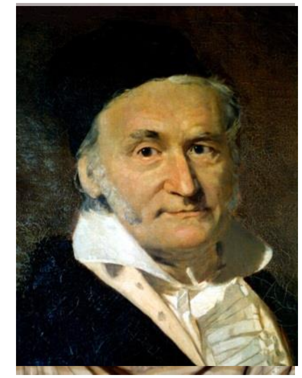
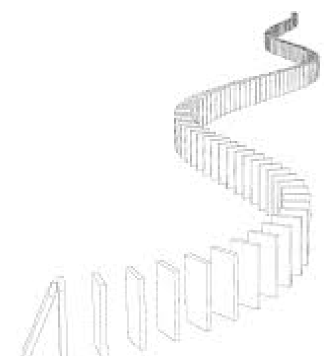


COMP 1002

Logic for Computer Scientists

Lecture 27

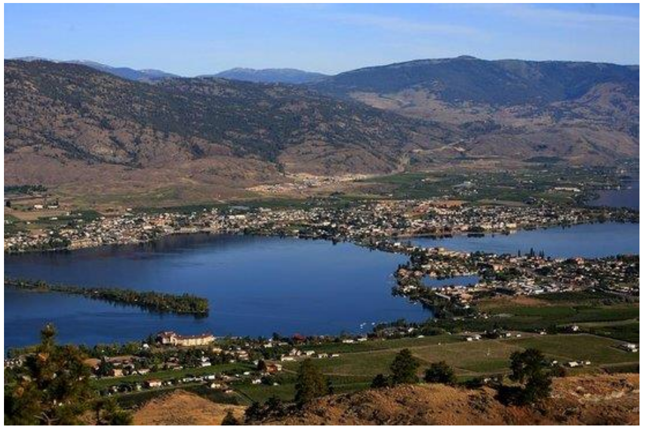


Puzzle: misspelling OSOYOOS

- In the game of Scrabble, players make words out of the pieces they have.
  - Suppose that someone puts the word "OSOYOOS" on the board, using up all her pieces.
  - How many ways could she have had the letters arranged on the rack in front of them?
    - The order of multiple copies of a letter does not matter: switching two S around results in the same sequence, but switching O and S does not.
    - The letters on the rack do not have to form a word.

Puzzle: misspelling OSOYOOS

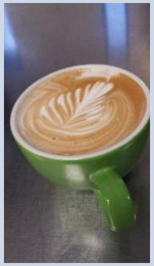
- Suppose that someone puts the word "OSOYOOS" on the board, using up all her pieces.
- How many ways could she have had the letters arranged on the rack in front of them?
  - There are 7 letters in the word OSOYOOS. If they were all distinct, that would be 7! = 5040 ways.
  - But there are 4 Os, and 2 Ss, order of which does not matter.
  - There are 4! ways to order Os, and 2! ways to order Ss.
  - Therefore, the total number of ways to order the letters ignoring the order of Os and Ss is ${}^{7!}/_{4!2!}$ = 105



Puzzle: misspelling OSOYOOS

- Suppose that someone puts the word “OSOYOOS” on the board, using up all her pieces.
- How many ways could she have had the letters arranged on the rack in front of them, *such that Ss are not next to each other*?
  - First, let’s consider all possible orderings of remaining letters: 5!/4! of them.
  - Now, consider places where S can go: _o_o_y_o_o_ (here, ooyoo are in arbitrary order). There are 6 such places.
  - So there are $\binom{6}{2} = {}^{6!}/_{2!4!}$ ways to place Ss.
  - Therefore, the total number of ways to order the letters ignoring the order of Os and Ss and with Ss not next to each other is ${}^{5!6!}/_{4!4!2!} = 75$
  - Alternatively, consider all orderings with Ss next to each other: there are $\frac{6!}{4!} = 30$ of them (treating the “SS” as a single letter).
  - Now, the total is 105-30 = 75.

Summary

Selecting k out of n objects	Order matters (permutations)	Order ignored (combinations)
With repetitions	n^k 	$\binom{k+n-1}{k}$ 
Without repetitions	$P(n,k) = \frac{n!}{(n-k)!}$ 	$\binom{n}{k}$ 

Binomial theorem

- Binomial expansion: open parentheses in $(x+y)^n$
- Open the parentheses in $(x+y)^2$
  - $x^2+2xy+y^2$
- Open parentheses in $(x+y)^3$
  - $x^3+xxy+xyx+yxx+xyy+yxy+yyx+y^3$
    $$= x^3+3x^2y+3xy^2+y^3$$
  - That is, a coefficient in front of x^2y is the number of ways to pick one y (or 2 x) out of 3 positions.
  - Call these coefficients **binomial coefficients**.
- B**inomial theorem**
  $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$
- Corollary: $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

Pascal's identity and triangle

- How to compute binomial coefficients?
  - First, note only need to compute them for $0 \le k \le \lceil \frac{n}{2} \rceil$, since $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$

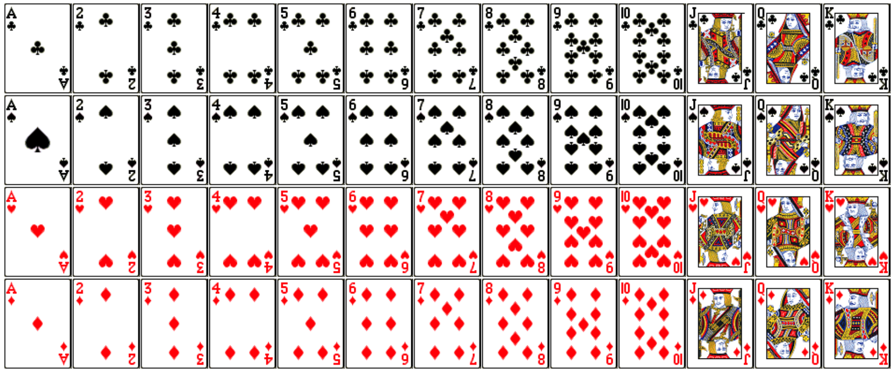
- Pascal's identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
  - In practice, use Stirling approximation
  - $n! \sim \sqrt{2\pi n}\,(n/e)^n$
  - So $\frac{\mathrm{n}^{\mathrm{k}}}{\mathrm{k}^{\mathrm{k}}} \le \binom{n}{k} < \frac{(en)^k}{k^k}$
  - And $\ln n! \sim n \ln n - n$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Pascal's triangle

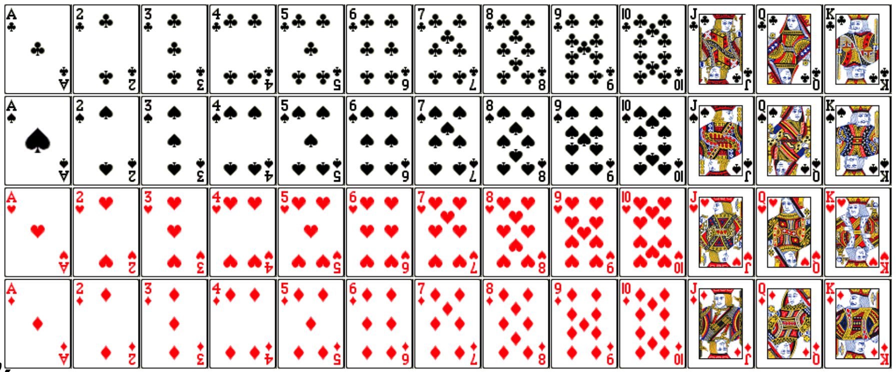
Puzzle: playing poker

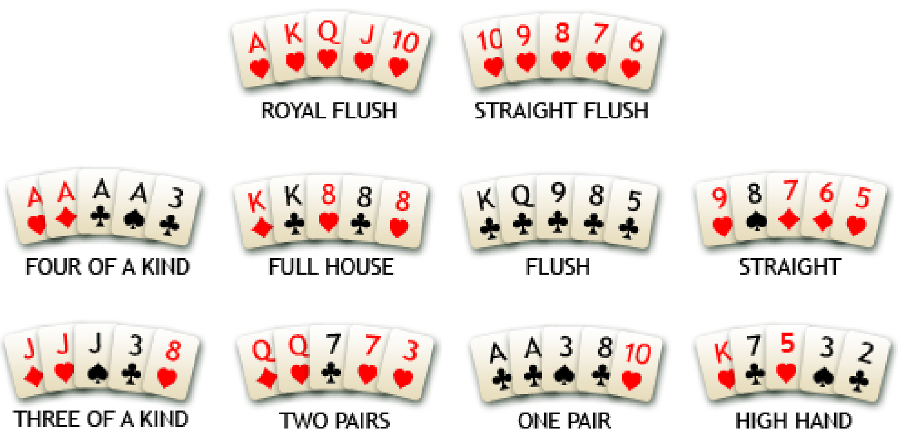
- There are 52 cards in a standard deck; 4 suites of 13 ranks each.
- In poker, some 5-card combinations (“hands”) are special:
  - For example, a “three of a kind” consists of three cards with the same rank, together with two arbitrary cards.
- How many ways are there to choose (ignoring the order)
  - a three of a kind hand?
  - A two pairs hand?
  - Other hands?...



Puzzle: playing poker

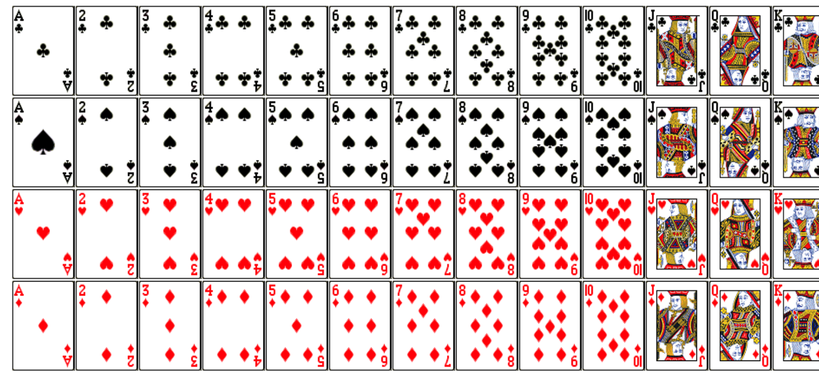
- There are 52 cards in a standard deck; 4 suites of 13 ranks each.
- In poker, some 5-card combinations (“hands”) are special:
  - For example, a “three of a kind” consists of three cards with the same rank, together with two cards of other different ranks.
- How many ways are there to choose (ignoring the order)
  - A royal flush?
  - a three of a kind hand?
  - a two pairs hand?
  - other hands?...





Puzzle: playing poker

- How many ways are there to choose (ignoring the order)
  - a royal flush?
    - C(4,1) = 4
  - a three of a kind?
    - pick the rank: 13=C(13,1)
    - Pick 3 out of 4 kinds of this rank: 4=C(4,3)
    - Pick two other ranks: C(12,2)= 66
    - Pick a suite of each of the other ranks: C(4,1)*C(4,1)=16
    - Total: 13*4*66*16=54912

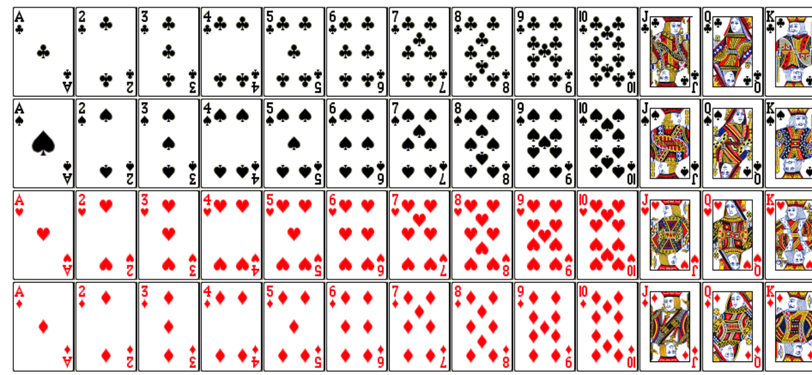


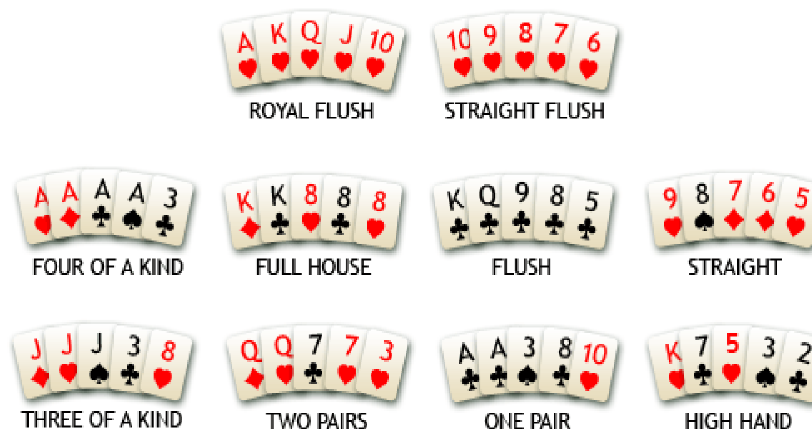
Finite probability

- More common: use the language of probability.
- **Experiments**: producing an **outcome** out of possible choices
  - Tossing a coin: outcome can be "heads"
  - Getting a lottery ticket: outcome can be "win"
- **Sample space S**: set of all possible outcomes.
  - {heads, tails} for coin toss
  - {1,2,3,4,5,6} x {1,2,3,4,5,6} for rolling two dice
- **Event A** $\subseteq \boldsymbol{S}$: subset of outcomes
  - Both dice came up even.
- **Probability** of an event if all outcomes are **equally likely**:
  - Pr(A)= |A|/|S| (fraction of the outcomes that are in the event A).
  - Probability of both dice coming up even:
    - A={ (2,2),(2,4),(4,2),(2,6),(6,2),(4,4), (4,6), (6,4) , (6,6)}. |A| =9, |S|=36
    - P(A)=9/36=1/4
- Can use the same combinatorics we just studied to calculate probabilities (i.e., for finding the size of A).

Puzzle: playing poker

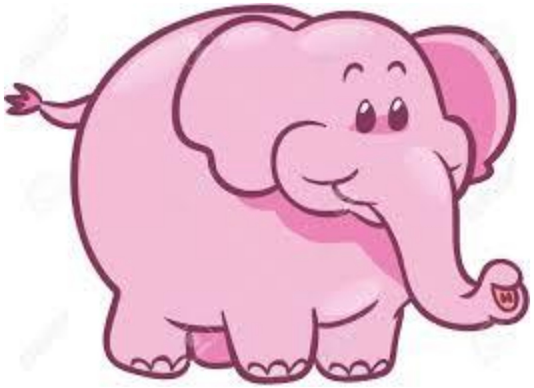
- What is the probability of getting a three of a kind hand?
  - Size of the sample space:
    - C(52, 5) = $\binom{52}{5} = 2{,}598{,}962$
  - Size of the event A:
    - 54,912
  - Probability of A:
    - Pr(A) = $\frac{|A|}{|S|} = 0.0211..$





Probabilities and pink elephants

- What is the probability that walking down George street you'd see a pink elephant?
  - Your friend says: "It is ½! You will either see the pink elephant, or not!"
    - Do you agree?

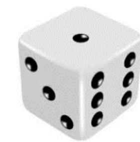


Probabilities and distributions

- What if outcomes are not equally likely?
  - Biased coins, etc.
- A function $Pr: S \to \mathbb{R}$ is a **probability distribution** on (a finite set) S if Pr satisfies the following:
  - For any outcome s $\in S,\ 0 \leq \Pr(s) \leq 1$
  - $\Sigma_{\{s \in S\}} \Pr(s) = 1$
- **Uniform distribution**: for all $s \in S, \Pr(s) = 1/|S|$
  - all outcomes are equally likely
  - Fair coin: $\Pr(heads) = \Pr(tails) = \frac{1}{2}$
- Biased coin: say heads twice as likely as tails.
  - $\Pr(heads) + \Pr(tails) = 1. \Pr(heads) = 2 * \Pr(tails)$
  - So $\Pr(heads) = \frac{2}{3}, \Pr(tails) = \frac{1}{3}$

Probabilities of events

- Probability of an event A is a sum of probabilities of the outcomes in A:
  - Pr(A)=$\Sigma_{\{a \in A\}} \Pr(a)$



- Probability of A not occurring:
  - $\Pr(\bar{A}) = 1 - \Pr(A)$

- Probability of the union of two events (either A or B happens) is

$$\Pr(A \cup \mathrm{B}) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

    - By principle of inclusion-exclusion
  - If A and B are disjoint, then $\Pr(A \cap B) = 0$, so $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

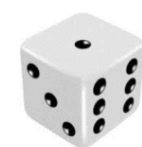
- In general, if events $A_1 \ldots A_n$ are pairwise disjoint
    - that is, $\forall i, j$ if $i \neq j$ then $A_i \cap A_j = \emptyset$
  - Then Pr($\cup_{i=1}^{n} A_i) = \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \Sigma_{i=1}^{n} \Pr(A_i)$
    - That is, probability of that any of the events happens is the sum of their individual probabilities.

Probabilities of events

- Suppose a die is biased so that 3 appears twice as often as any other number (others equally likely).
  - Probability of 3: 2/7. Probabilities of others: 1/7

- What is the probability that an odd number appears?
  - Event: A={1,3,5}
  - Pr(A) = 1/7+2/7+1/7=4/7.

- What is a probability that either an odd number or a number divisible by 3 appears?
  - A={1,3,5}. B = {3,6} . $A \cap B = \{3\}$
  - Pr(A) = 4/7. Pr(B) = 3/7. $\Pr(A \cap B) = 2/7$
  - $\Pr(A \cup B) = \Pr(\{1,3,5,6\})$



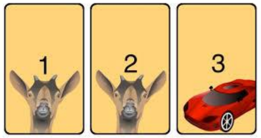
$$= \frac{4}{7} + \frac{3}{7} - \frac{2}{7} = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} + \frac{1}{7} = \frac{5}{7}$$

Birthday paradox

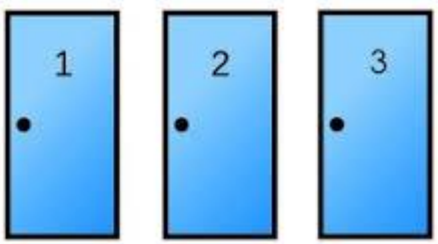
- How many people have to be in the room so that probability that two of them have the same birthday is at least ½?

Birthday paradox

- How many people have to be in the room so that probability that two of them have the same birthday is at least ½?
  - Considering all birthdays independent: no twins!
  - And considering all days equally likely
    - Otherwise probability would be higher.
  - Even counting leap years: 366 days.
- Product rule: number of combinations of distinct birthdays of the first i people is P(i, 366) = 366*365*…*(366-i+1)
  - Probability that the first i people all have different birthday is $\frac{P(i,366)}{366^i} = \frac{365}{366}\frac{364}{366} \dots \frac{(366-i+1)}{366}$
  - So with probability $1 - \frac{P(i,366)}{366^i}$ at least two out of first i people have birthday on the same day.
  - That comes up to about i =23 people to reach ½.



Puzzle: Monty Hall problem



- Let's make a deal!
  - A player picks a door.
  - Behind one door is a car.
  - Behind two others are goats.
- A player chooses a door.
  - A host opens another door
  - Shows a goat behind it.
  - And asks the player if she wants to change her choice.
- Should she switch?