



COMP 1002

Logic for Computer Scientists

Lecture 26





Puzzle: chocolate squares



- Suppose you have a piece of chocolate like this:



- How many squares are in it?
 - of all sizes,
 - from single to the whole thing

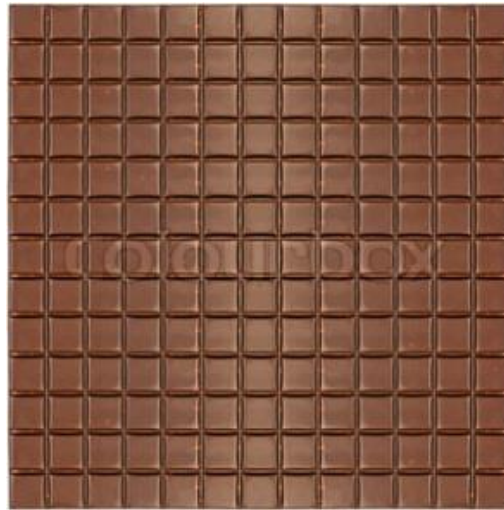
1. One square 4x4
 2. Four squares 3x3
 - Can start (e.g, top-left corner) at $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$
 - 2 choices for a row, 2 choices for a column.
 3. Nine squares 2x2
 - Can start at any (x,y) with $x \in \{1,2,3\}$, $y \in \{1,2,3\}$
 - So $3*3 = 9$ choices.
 4. 16 squares 1x1.
- Total: $1+4+9+16=30$ squares



Puzzle: chocolate squares



- Suppose you have a piece of chocolate like this:



- How many squares are in it?
 - of all sizes,
 - from single to the whole thing

1. One square 13x13
2. ...
3. $13 \times 13 = 169$ squares 1x1
 - 12×12 2x2
 - 11×11 3x3
 -

General formula: starting with an $n \times n$ piece of chocolate, get

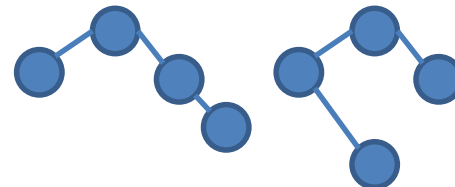
$$\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$$

Total number of squares.



Combinatorics

- Counting various ways to arrange things.
 - Leading to probability theory.
- How long would a program that does brute-force search over possibilities will run?
 - Depends on the number of potential answers.
 - How many ways a sorting algorithm can rearrange its n-element input?
- How many possibilities need to be checked to break 4-digit PIN? A 8-letter password (with digits and special symbols)? A 80-letter passphrase (with just letters)? Which one is more secure?
 - 10 digits + 23 special symbols + 26 lowercase + 26 uppercase letters.
- What if some of the possibilities are identical to each other? How can we count then?
 - How many different trees are there, if two trees are considered the same if one can be transformed into another by moving/renameing vertices, keeping edges attached.
 - Flip and move bottom vertex.





Rules of sum and product

- Rule of sum:
 - If there are n choices for A , and m choices for B , then there are $n+m$ choices for “ A or B ”
 - Provided A and B do not overlap.
 - If there are 16 squares of size 1, and 9 squares of size 2, then there are 25 squares of size either 1 or 2.
- Rule of product:
 - If there are n choices for A , and m choices for B , then there are $n*m$ choices for “ A and B ”.
 - 3 choices for a row times 3 choices for a column: 9 of 2×2 squares.
 - Can also count rectangles rather than squares...



Cartesian products

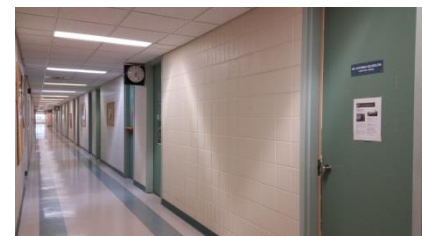
- When a sequence is a Cartesian product of n copies of the same set:
 - How many possible PINs consisting of 4 digits are there?
 - Using the rule of product. Each of the digits has 10 possibilities (0...9), and picking a digit for one position does not affect others.
 $10*10*10*10=10^4 = 10,000$.
 - So by the Pigeon Hole Principle, there are (lots of) people at MUN that have the same PIN.
 - How many rows does a truth table on n variables have?
 - Each variable has 2 possibilities. So 2^n .
 - In general, if there are n independent places in the sequence, and m possibilities for each place, get m^n possible sequences.
- When a sequence is a Cartesian product of different sets:
 - Multiply together sizes of these sets.
 - How many different dishes can you make out of: 3 types of protein (meat, chicken, falafel), 2 types of starch (noodles, rice), 4 types of sauces?
 - $3*2*4$.





Permutations

- **Permutations:** number of sequences of objects.
 - Without repetition: each object appears once.
- Example: how many ways to assign offices ER-6030 to ER-6033 to Antonina, Dave, Sharene and Yuanzhu?
 - 4 choices to pick who gets ER-6030. This leaves 3 choices to pick who gets ER-6031. Now 2 remain for ER-6032, and the last one is stuck with ER-6033.
 - By the product rule, get $4*3*2*1 = 4!$
- In general, number of permutations of n elements is $n!$
 - “Permutations”: the difference between choices is only the order of elements.





r-Permutations.

- Before, we talked about permutations of all objects in the set. An **r-permutation** $P(n,r)$ involves taking only r out of n objects, and counting the number of possible sequences.
 - That is, the task consists of 1) picking r out of n objects. 2) counting number of sequences out of them.
- Example:
 - How many ways to assign ER offices to 4 out of 25 faculty members?
 - Pick one out of 25 to be in ER-6030. One out of remaining 24 to be in ER-6031, one out of 23 for ER-6032, one out of 22 for ER-6033. By the product rule, get $P(25,4) = 25*24*23*22$.
 - Alternative way of thinking:
 - There are $25!$ ways to assign offices to everybody.
 - Out of them, $21!$ way to assign non-ER offices. We are not interested in what everybody else got – so once the way to assign ER offices is fixed, all sequences with this assignment to ER offices are the same for us.
 - Get $P(25,4) = \frac{25!}{21!} = 25*24*23*22$ possible ways to get 4 people offices in ER.
- General formula:
$$P(n, r) = \frac{n!}{(n-r)!}$$



Combinations.

- In general, how many ways to pick k out of n objects?
 - The number of k -permutations divided by the number of different permutations of the k objects themselves.
- **Combinations:** $C(n,k)$ is number of ways to choose k objects out of n .

- “ n choose k ”: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = C(n,k)$

- How many ways to choose 4 out of 25 faculty members to get an office in Earth Science?

- 25! ways to assign offices altogether.
- 4! ways to assign the first 4 offices (ones in Earth Science).
- 21! ways to assign offices not in Earth Science.
- Overall, $C(25,4) = \frac{25!}{4! 21!}$ ways to pick who gets an Earth Science office.

- How many ways to select a crew of 6 astronauts out of a team of 30 to go to Mars?

- $C(30,6) = \binom{30}{6} = \frac{30!}{6!(24!)} = 593,775$





Combinations with repetition







- Suppose that 10 of you came to the office hour, and we decided to go to Jumping Bean to get some coffee.
- Jumping Bean sells 6 types of coffee drinks: drip coffee, cappuccino, espresso, latte, mocha and americano.
- How many different orders can we place, if each gets one coffee drink?
 - That is, how many ways are there to select 11 items, where each item comes from one of the 6 categories?
 - Let's use one letter for each type:
 - e.g., aacdeeellmm stands for 2 americanos, one cappuccino, one drip, three espressos, 2 lattes and 2 moccas.
 - Idea: think of a string of letters with “dividers” between different types of drinks:
 - aa|c|d|eee||l|mm
 - How many ways are there to position the dividers?
 - $6-1=5$ dividers
 - Number of orders + number of dividers: $11+6-1=16$ positions.
 - So $\binom{16}{11} = \binom{16}{5} = 4368$ possible orders.
- In general, number of ways to select r objects out of n categories with repetition is $\binom{r+n-1}{r}$





Summary

Selecting k out of n objects	Order matters (permutations)	Order ignored (combinations)
With repetitions	n^k 	$\binom{k+n-1}{k}$ 
Without repetitions	$P(n, k) = \frac{n!}{(n-k)!}$ 	$\binom{n}{k}$ 

Puzzle: misspelling OSOYOOS

- In the game of Scrabble, players make words out of the pieces they have.
 - Suppose that someone puts the word “OSOYOOS” on the board, using up all her pieces.
 - How many ways could she have had the letters arranged on the rack in front of them?
 - The order of multiple copies of a letter does not matter: switching two S around results in the same sequence, but switching O and S does not.
 - The letters on the rack do not have to form a word.

