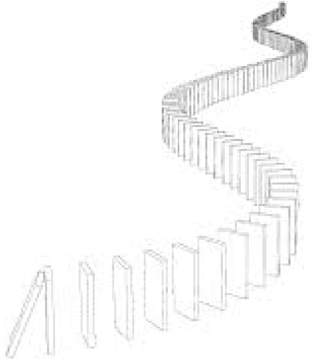
# COMP 1002

# Logic for Computer Scientists

## **Lecture 25**

# Puzzle

- Do the following two English sentences have the same parse trees?
  - Time flies like an arrow.

  - Fruit flies like an apple.

# Recursive definitions of sets

- **Basis:** initial elements in the set
  a) Empty string is in S.
  b) Empty string, 0 and 1 are in S.
- **Recursive step:** a rule to make new elements in the set out of existing ones in S
  a) if $w \in S$, then $w0 \in S$ and $w1 \in S$
  b) If $w_1 \in S$ and $w_2 \in S$, then $w_1 w_2 \in S$
- **Restriction:** and nothing else is in S.
  – Nothing else is a binary string.

- Both examples define a set of all binary strings $\{0,1\}^*$
  – Recursive step b) only works with basis b).
  – Recursive step a) works with both basis a) and basis b).

# Structural induction

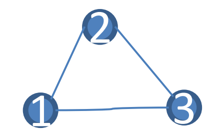
- Let $S \subseteq U$ be a recursively defined set, and F(x) is a property (of $x \in U$).
- Then
  - if all $x$ in the base of S have the property,
  - and applying the recursion rules preserves the property,
  - then all elements in S have the property.

# Multiples of 3

- Let's define a set S of numbers as follows.
  - Base: $3 \in S$
  - Recursion: if $x, y \in S$, then $x + y \in S$
- Claim: all numbers in S are divisible by 3
  - That is, $\forall x \in S\ \exists\ z \in \mathbb{N}\ x = 3z$.
- Proof (by structural induction).
  - **Base case:** 3 is divisible by 3 (z=1).
  - **Recursive step:**
    - Let $x, y \in S$. Then $\exists z, u \in \mathbb{N}\ x = 3z \wedge y = 3u$. **(inductive hypothesis)**
    - Then $x + y = 3z + 3u = 3(z + u)$. **(induction step)**
    - Therefore, $x + y$ is divisible by 3.
  - As there are no other elements in S except for those constructed from 3 by the recursion rule, all elements in S are divisible by 3.

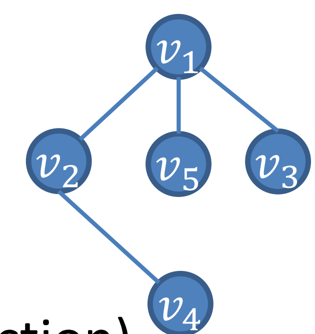
# Trees

- In computer science, a **tree** is an undirected graph without cycles
  - **Undirected**: all edges go both ways, no arrows.
  - **Cycle**: sequence of edges going back to the same point.



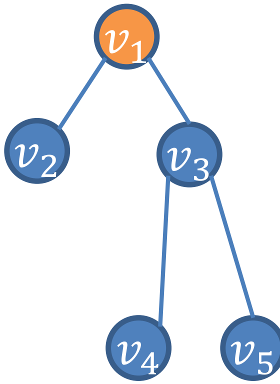
Undirected cycle (not a tree)

- Recursive definition of trees:
  - Base: A single vertex $v$ is a tree.
  - Recursion:
    - Let $T$ be a tree, and $v$ a new vertex.
    - Then a new tree consist of $T$, $v$, and an edge (connection) between some vertex of $T$ and $v$.
  - Restriction:
    - Anything that cannot be constructed with this rule from this base is not a tree.



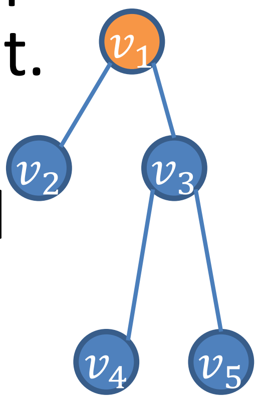
# Binary trees

- **Rooted trees** are trees with a special vertex designated as a root.
  - Rooted trees are **binary** if every vertex has at most three edges: one going towards the root, and two going away from the root. **Full** if every vertex has either 2 or 0 edges going away from the root.
- Recursive definition of full binary trees:
  - Base: A single vertex $v$ is a full binary tree with that vertex as a root.
  - Recursion:
    - Let $T_1, T_2$ be full binary trees with roots $r_1, r_2$, respectively. Let $v$ be a new vertex.
    - A new full binary tree with root $v$ is formed by connecting $r_1$ and $r_2$ to $v$.
  - Restriction:
    - Anything that cannot be constructed with this rule from this base is not a full binary tree.



# Height of a full binary tree

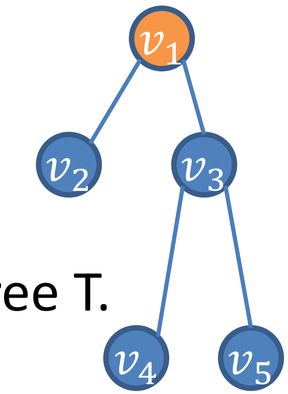
- The **height** of a rooted tree, $h(T)$, is the maximum number of edges to get from any vertex to the root.
  - Height of a tree with a single vertex is 0.
- Claim: Let $n(T)$ be the number of vertices in a full binary tree T. Then $n(T) \leq 2^{h(T)+1} - 1$



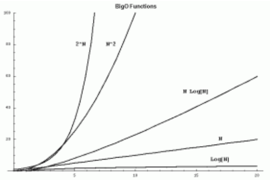

Height 2

- Alternatively, height of a binary tree is at least $\log_2 n(T)$
  - If you have a recursive program that calls itself twice (e.g, within if … then … else …)
  - Then if this code executes n times (maybe on n different cases)
  - Then the program runs in time at least $\log_2 n$, even when cases are checked in parallel.

# Height of a full binary tree



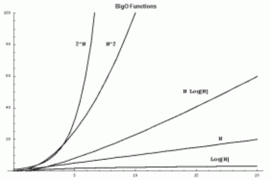

- Claim: Let $n(T)$ be the number of vertices in a full binary tree T. Then $n(T) \leq 2^{h(T)+1} - 1$, where $h(T)$ is the height of T.
- Proof (by structural induction)
  - Base case: a tree with a single vertex has $n(T) = 1$ and $h(T) = 0$.
    - So $2^{h(T)+1} - 1 = 1 \geq 1$
  - Recursion: Suppose $T$ was built by attaching $T_1, T_2$ to a new root vertex $v$.
    - Number of vertices in $T$ is $\mathrm{n(T)} = n(T_1) + n(T_2) + 1$
    - Every vertex in $T_1$ or $T_2$ now has one extra step to get to the new root in $T$.
      - So $h(T) = 1 + \max(h(T_1), h(T_2))$
    - By the induction hypothesis, $n(T_1) \leq 2^{h(T_1)+1} - 1$ and $n(T_2) \leq 2^{h(T_2)+1} - 1$
    - $$\begin{aligned} \mathrm{n(T)} &= n(T_1) + n(T_2) + 1 \\ &\leq 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) \\ &\leq 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 \\ &\leq 2 \cdot 2^{\max(h(T_1), h(T_2))+1} - 1 \\ &= 2 \cdot 2^{h(T)} - 1 = 2^{h(T)+1} - 1 \end{aligned}$$
  - Therefore, the number of vertices of any binary tree $T$ is $\leq 2^{h(T)+1} - 1$




# Function growth.

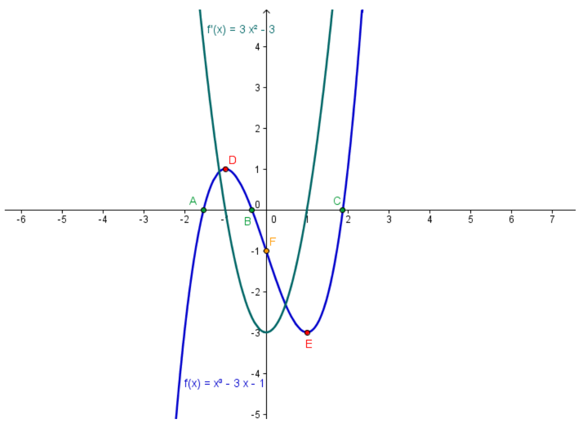
- What does it mean to “grow” at a certain speed? How to compare growth rate of two functions?
  - Is f(n)=100n larger than $g(n) = n^2$?
    - For small n, yes. For n > 100, not so much…
  - As usually program take longer on larger inputs, performance on larger inputs matters more.
  - Constant factors don’t matter that much.

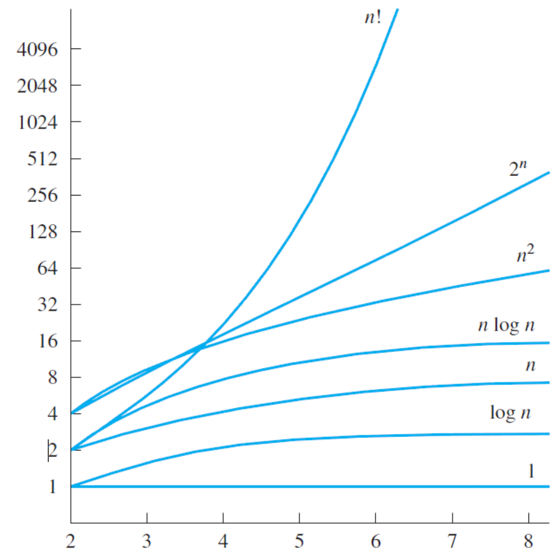
- So to compare two functions, check which becomes larger as n increases (to infinity).
  - Ignoring constant factors, as they don’t contribute to the rate of growth.



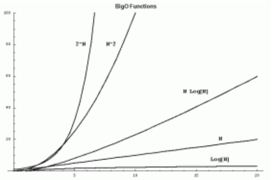
# Function growth.

- How to estimate the rate of growth?
  – Plotting a graph?




- Not quite conclusive:
  – How do you know what they will do past the graphed part?



# O-notation.

- We say that f(n) grows at most as fast as g(n) if
  - There is a value $n_0$ such that after $n_0$, $\mathrm{f}(n)$ is always at most as large as $\mathrm{g}(n)$
    - More precisely, compare absolute values: |g(n)| vs. |f(n)|
  - Moreover, ignore constant factors:
    - So if two functions only differ by a constant factor, consider them having the same growth rate.

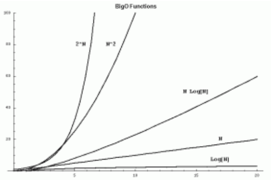
- Denote set of all functions growing at most as fast as $g(n)$ by $\boldsymbol{O\big(g(n)\big)}$
  - **Big-Oh** of g(n).
  - g(n) is an **asymptotic upper bound** for f(n).
  - When both $f(n) \in O\big(g(n)\big)$ and $g(n) \in O\big(f(n)\big)$, write $f(n) \in \Theta(g(n))$
    - f(n) is in **big-Theta** of g(n)).

- More generally, for real-valued functions f(x) and g(x),

$$f(x) \in O\big(g(x)\big) \text{ iff}$$

$$\exists\, x_0 \in \mathbb{R}^{\geq 0}\ \exists c \in \mathbb{R}^{>0}\ \ \forall x \geq x_0\ \ |f(x)| \leq c \cdot |g(x)|$$

- That is, from some point $x_0$ on, each $|f(x)|$ is less than |g(x)| (up to a constant factor).
- Usually in time complexity have functions $\mathbb{N} \to \mathbb{R}^{\geq 0}$, so use $n$ for $x$ and ignore | |.



# O-notation.

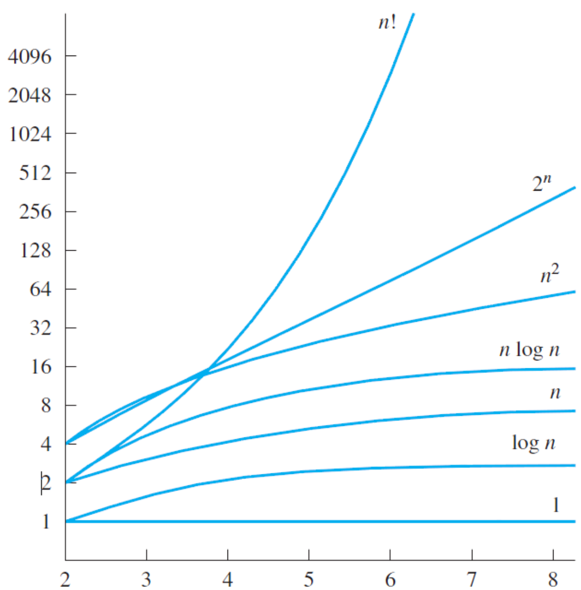
$f(n) \in O(g(n))$ iff

$$\exists\, n_0 \in \mathbb{N}\ \exists c \in \mathbb{R}^{>0}\ \ \forall n \geq n_0\ \ f(n) \leq c \cdot g(n)$$

- $f(n) = n^2, g(n) = 2^n$.
  - Take c=1, $n_0 = 4$.
  - For every $n \geq n_0, f(n) \leq g(n)$
    - Proof by induction.
  - So $\mathrm{n}^2 \in O(2^n)$

- $f(n) = n^2, g(n) = 10n$.
  - Take arbitrary $c$ and look at $n^2 \leq c \cdot 10n$.
  - No matter what $c$ is, when $n > c \cdot 10,\ n^2 \geq c \cdot 10n$
  - So $n^2 \notin O(10n)$.

- $f(n) = n^2 + 100n, g(n) = 10n^2$.
  - Here, $f(n) \in O(g(n))$ and also $g(n) \in O(f(n))$
    - So $f(n) \in \Theta(g(n))$
    - $f(n) \in O(g(n))$: c = 20 and/or $n_0 = 100$ work.
    - $g(n) \in O(f(n))$: Take c=10, $n_0 = 1$.
  - Can ignore not only constants, but also all except the leading term in the expression.




You will see some O-notation in COMP 1000 and a lot in COMP 2002.

# Tower of Hanoi game

- Rules of the game:
  - Start with all disks on the first peg.
  - At any step, can move a disk to another peg, as long as it is not placed on top of a smaller disk.
  - Goal:  move the whole tower onto the second peg.
- Question:  how many steps are needed to move the tower of 8 disks? How about n disks?

# Tower of Hanoi game


- Rules of the game:
  - Start with all disks on the first peg.
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  - Goal: move the whole tower onto the second peg.
- Question: how many steps are needed to move the tower of 8 disks? How about n disks?

- Let us call the number of moves needed to transfer n disks H(n).
  - Names of pegs do not matter: from any peg $i$ to any peg $j \neq i$ would take the same number of steps.
- Basis: only one disk can be transferred in one step.
  - So H(1) = 1
- Recursive step:
  - suppose we have n-1 disks. To transfer them all to peg 2, need $H(n-1)$ number of steps.
  - To transfer the remaining disk to peg 3, 1 step.
  - To transfer n-1 disks from peg 2 to peg 3 need H(n-1) steps again.
  - So H(n) = 2H(n-1)+1 (recurrence).
- Closed form: H(n) = $2^n - 1$.

# Recurrence relations

- **Recurrence**: an equation that defines an $n^{th}$ element in a sequence in terms of one or more of previous terms.
    - Think of $F(n) = s_n$ for some sequence $\{s_n\}$
  - $H(n) = 2H(n-1) + 1$
  - $F(n) = F(n-1) + F(n-2)$

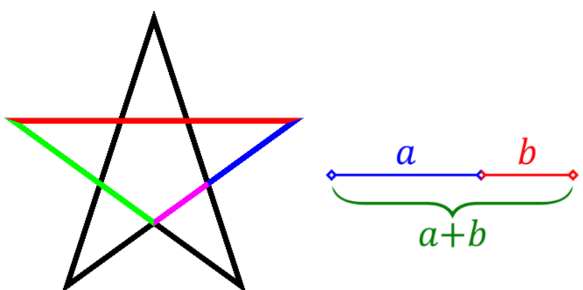
- A **closed form** of a recurrence relation is an expression that defines an $n^{th}$ element in a sequence in terms of $n$ directly.
  - Often use recurrence relations and their closed forms to describe performance of (especially recursive) algorithms.

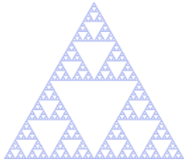
# Closed forms of some sequences

- Arithmetic progression:
  - Sequence: $c, c+d, c+2d, c+3d, \ldots, c+nd, \ldots$
  - **Closed form:** $s_n = c + nd$
    - Closed forms are very useful for analysis of recursive programs, etc.

- Geometric progression:
  - Sequence: $c, cr, cr^2, cr^3, \ldots, cr^n, \ldots$
  - **Closed form:** $s_n = c \cdot r^n$

- Fibonacci sequence: F(n)=F(n-1)+F(n-2)
  - Sequence: 1,1,2,3,5,8,13, ...
  - **Closed form:** $F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$
    - Where $\varphi$ *("phi")* is the "golden ratio": a ratio such that $\frac{a+b}{a} = \frac{a}{b}$
    - $\varphi = \frac{1+\sqrt{5}}{2}$





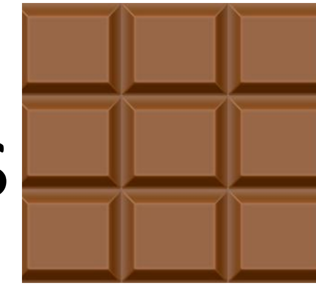
# Solving recurrences

- Solving a recurrence: finding a closed form.
  - Solving  the recurrence H(n)=2H(n-1)+1
    - H(n) $= 2 \cdot H(n-1) + 1$

      $$= 2(2H(n-2)+1)+1 = 2^2H(n-2)+2+1$$
      $$= 2^3H(n-3)+2^2+2+1$$
      $$= 2^4\,H(n-4)+2^3+2^2+2+1 \ldots$$

  - Closed form: $H(n) = \Sigma_{i=0}^{n-1} 2^i = 2^n - 1$
    - Proof by induction (coming in the next lecture).
    - Or by noticing that a binary number 111...1 plus 1 gives a binary number 10000…0
  - So adding one more disk doubles the number of steps.
    - We say that the function defined by H(n) **grows exponentially**
- Solving recurrences in general might be tricky.
  - When the recurrence is of the form T(n)=a T(n/b)+f(n), there is a  general method to estimate the growth rate of a function defined by the recurrence
    - Called the Master Theorem for recurrences.

# Puzzle: chocolate squares



- Suppose you have a piece of chocolate like this:

- How many squares are in it?
  – of all sizes, from single to the whole thing